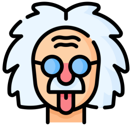
Metric learning from lazy, opinionated crowds

i.e., from limited pairwise preference comparisons

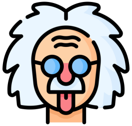
Geelon So, agso@ucsd.edu
EnCORE Student Social — February 26, 2024

An opinionated member of society



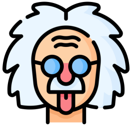
An opinionated member of society

I prefer Blade Runner over Godzilla.



An opinionated member of society

I prefer Blade Runner over Godzilla.



For it is more similar to my favorite movie The Matrix.

Metric learning from preferences

Question:
Suppose a lot of people on the internet tell us these sorts of pairwise movie rankings.

Metric learning from preferences

Question:
Suppose a lot of people on the internet tell us these sorts of pairwise movie rankings.

- Can we learn a metric that captures the similarity of movies in general?

Background

Metric learning: raison d'être

Distance-based algorithms

- nearest neighbor methods
- margin-based classification
- information retrieval
- clustering
- etc.

Metric learning: raison d'être

Distance-based algorithms

- nearest neighbor methods
- margin-based classification
- information retrieval
- clustering
- etc.

- The behavior/performance are often sensitive to the choice of distance.

Metric learning: raison d'être

Distance-based algorithms

- nearest neighbor methods
- margin-based classification
- information retrieval
- clustering
- etc.

- The behavior/performance are often sensitive to the choice of distance.
- Good metrics (e.g. for visual similarity) are hard to construct by hand.

Metric learning: raison d'être

Distance-based algorithms

- nearest neighbor methods
- margin-based classification
- information retrieval
- clustering
- etc.

- The behavior/performance are often sensitive to the choice of distance.
- Good metrics (e.g. for visual similarity) are hard to construct by hand.

Goal of metric learning:
Automatically learn a good metric for these downstream tasks

Metric learning: raison d'être

Distance-based algorithms

- nearest neighbor methods
- margin-based classification
- information retrieval
- clustering
- etc.

- The behavior/performance are often sensitive to the choice of distance.
- Good metrics (e.g. for visual similarity) are hard to construct by hand.

Goal of metric learning:
Automatically learn a good metric for these downstream tasks

- esp. metrics aligning with human values, perception, and preferences.

The alignment problem

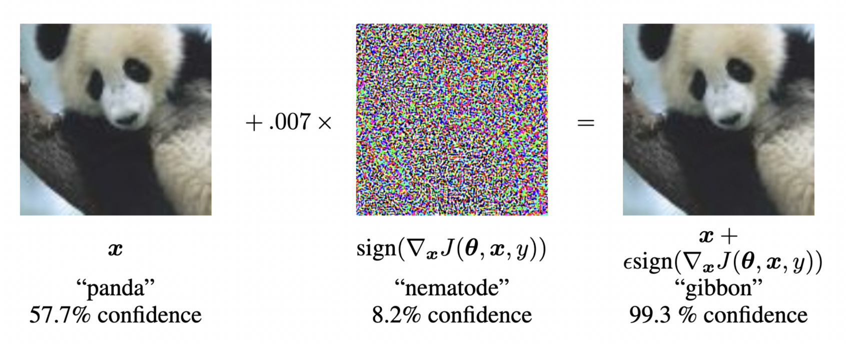


Figure 1: These two images are visually indistinguishable to a human, but very well-separated under the Euclidean distance (Goodfellow et al., 2014).

Related work

Metric learning from triplet comparisons

Related work

Metric learning from triplet comparisons

- Goal: given a set of items $\mathcal{X}$, learn a metric ρ over the items.

Related work

Metric learning from triplet comparisons

- Goal: given a set of items $\mathcal{X}$, learn a metric ρ over the items.
- Feedback: "A is more similar to B than to C."

Related work

Metric learning from triplet comparisons

- Goal: given a set of items $\mathcal{X}$, learn a metric ρ over the items.
- Feedback: "A is more similar to B than to C."

Schultz and Joachims (2003), Verma and Branson (2015), Mason et al. (2017)

Related work

Metric learning from triplet comparisons

- Goal: given a set of items $\mathcal{X}$, learn a metric ρ over the items.
- Feedback: "A is more similar to B than to C."

Schultz and Joachims (2003), Verma and Branson (2015), Mason et al. (2017)

Simultaneous metric and preference learning

Related work

Metric learning from triplet comparisons

- Goal: given a set of items $\mathcal{X}$, learn a metric ρ over the items.
- Feedback: "A is more similar to B than to C."

Schultz and Joachims (2003), Verma and Branson (2015), Mason et al. (2017)

Simultaneous metric and preference learning

- Assumption: a user has an *ideal item* A and prefers items more similar to A.

Related work

Metric learning from triplet comparisons

- Goal: given a set of items $\mathcal{X}$, learn a metric ρ over the items.
- Feedback: "A is more similar to B than to C."

Schultz and Joachims (2003), Verma and Branson (2015), Mason et al. (2017)

Simultaneous metric and preference learning

- Assumption: a user has an *ideal item* A and prefers items more similar to A.
- Goal: given a set of items $\mathcal{X}$, learn ideal item A and metric ρ capturing similarity.

Related work

Metric learning from triplet comparisons

- Goal: given a set of items $\mathcal{X}$, learn a metric ρ over the items.
- Feedback: "A is more similar to B than to C."

Schultz and Joachims (2003), Verma and Branson (2015), Mason et al. (2017)

Simultaneous metric and preference learning

- Assumption: a user has an *ideal item* A and prefers items more similar to A.
- Goal: given a set of items $\mathcal{X}$, learn ideal item A and metric ρ capturing similarity.
- Feedback: "I prefer B over C."

Related work

Metric learning from triplet comparisons

- Goal: given a set of items $\mathcal{X}$, learn a metric ρ over the items.
- Feedback: "A is more similar to B than to C."

Schultz and Joachims (2003), Verma and Branson (2015), Mason et al. (2017)

Simultaneous metric and preference learning

- Assumption: a user has an *ideal item* A and prefers items more similar to A.
- Goal: given a set of items $\mathcal{X}$, learn ideal item A and metric ρ capturing similarity.
- Feedback: "I prefer B over C."
  - triplet comparison with a latent comparator (i.e. A is not observed)

Related work

Metric learning from triplet comparisons

- Goal: given a set of items $\mathcal{X}$, learn a metric ρ over the items.
- Feedback: "A is more similar to B than to C."

Schultz and Joachims (2003), Verma and Branson (2015), Mason et al. (2017)

Simultaneous metric and preference learning

- Assumption: a user has an *ideal item* A and prefers items more similar to A.
- Goal: given a set of items $\mathcal{X}$, learn ideal item A and metric ρ capturing similarity.
- Feedback: "I prefer B over C."
    - triplet comparison with a latent comparator (i.e. A is not observed)
    - much more prevalent form of feedback than triplet comparisons

Related work

Metric learning from triplet comparisons

- Goal: given a set of items $\mathcal{X}$, learn a metric ρ over the items.
- Feedback: "A is more similar to B than to C."

Schultz and Joachims (2003), Verma and Branson (2015), Mason et al. (2017)

Simultaneous metric and preference learning

- Assumption: a user has an *ideal item* A and prefers items more similar to A.
- Goal: given a set of items $\mathcal{X}$, learn ideal item A and metric ρ capturing similarity.
- Feedback: "I prefer B over C."
  - triplet comparison with a latent comparator (i.e. A is not observed)
  - much more prevalent form of feedback than triplet comparisons

Xu and Davenport (2020) and Canal et al. (2022)

Our work in relationship

Let $\mathbb{R}^d$ be equipped with an unknown Mahalanobis distance ρ_M.

Our work in relationship

Let $\mathbb{R}^d$ be equipped with an unknown Mahalanobis distance ρ_M.

What is known:

- we can learn ρ_M using $\Theta(d^2)$ measurements from a single user

Our work in relationship

Let $\mathbb{R}^d$ be equipped with an unknown Mahalanobis distance ρ_M.

What is known:

- we can learn ρ_M using $\Theta(d^2)$ measurements from a single user
- or, using $\Theta(d)$ measurements from $\Omega(d)$ users

Our work in relationship

Let $\mathbb{R}^d$ be equipped with an unknown Mahalanobis distance ρ_M.

What is known:

- we can learn ρ_M using $\Theta(d^2)$ measurements from a single user
- or, using $\Theta(d)$ measurements from $\Omega(d)$ users
  - $\Theta(d)$ is necessary for simultaneous recovery of metric and ideal points

Our work in relationship

Let $\mathbb{R}^d$ be equipped with an unknown Mahalanobis distance ρ_M.

What is known:

- we can learn ρ_M using $\Theta(d^2)$ measurements from a single user
- or, using $\Theta(d)$ measurements from $\Omega(d)$ users
  - $\Theta(d)$ is necessary for simultaneous recovery of metric and ideal points

Limitations:

- modern representations of data can be extremely high dimensional

Our work in relationship

Let $\mathbb{R}^d$ be equipped with an unknown Mahalanobis distance ρ_M.

What is known:

- we can learn ρ_M using $\Theta(d^2)$ measurements from a single user
- or, using $\Theta(d)$ measurements from $\Omega(d)$ users
  - $\Theta(d)$ is necessary for simultaneous recovery of metric and ideal points

Limitations:

- modern representations of data can be extremely high dimensional
- it could be infeasible to obtain $\Theta(d)$ measurements per user

Our work in relationship

Let $\mathbb{R}^d$ be equipped with an unknown Mahalanobis distance ρ_M.

What is known:

- we can learn ρ_M using $\Theta(d^2)$ measurements from a single user
- or, using $\Theta(d)$ measurements from $\Omega(d)$ users
  - $\Theta(d)$ is necessary for simultaneous recovery of metric and ideal points

Limitations:

- modern representations of data can be extremely high dimensional
- it could be infeasible to obtain $\Theta(d)$ measurements per user

Our work: Let's just give up on trying to learn the ideal points. We ask:

Can we recover the metric using $m \ll d$ measurements per user?

Preliminaries

Formal setting

Representation space

Let $\mathcal{X}$ be a set of items embedded into $\mathbb{R}^d$.

- Assume $\mathbb{R}^d$ is equipped with an unknown Mahalanobis distance ρ_M.

Formal setting

Representation space

Let $\mathcal{X}$ be a set of items embedded into $\mathbb{R}^d$.

- Assume $\mathbb{R}^d$ is equipped with an unknown Mahalanobis distance ρ_M.

Feedback model

Users provide preference feedback under the *ideal point model* (Coombs, 1950).

Formal setting

Representation space

Let $\mathcal{X}$ be a set of items embedded into $\mathbb{R}^d$.

- Assume $\mathbb{R}^d$ is equipped with an unknown Mahalanobis distance ρ_M.

Feedback model

Users provide preference feedback under the *ideal point model* (Coombs, 1950).

- Assume each user has a ideal point $u \in \mathbb{R}^d$ (which we cannot observe).

Formal setting

Representation space

Let $\mathcal{X}$ be a set of items embedded into $\mathbb{R}^d$.

- Assume $\mathbb{R}^d$ is equipped with an unknown Mahalanobis distance ρ_M.

Feedback model

Users provide preference feedback under the *ideal point model* (Coombs, 1950).

- Assume each user has a ideal point $u \in \mathbb{R}^d$ (which we cannot observe).
- The user prefers an item x over x' whenever:

$$\rho_M(u, x) < \rho(u, x').$$

Formal setting

Representation space

Let $\mathcal{X}$ be a set of items embedded into $\mathbb{R}^d$.

- Assume $\mathbb{R}^d$ is equipped with an unknown Mahalanobis distance ρ_M.

Feedback model

Users provide preference feedback under the *ideal point model* (Coombs, 1950).

- Assume each user has a ideal point $u \in \mathbb{R}^d$ (which we cannot observe).
- The user prefers an item x over x' whenever:

$$\rho_M(u, x) < \rho(u, x').$$

- We receive measurements from users of the form:

$$(x, x', y) \qquad \text{where} \quad y = \mathbf{1}\{\rho_M(u, x) < \rho_M(u, x')\}.$$

Mahalanobis distances

A *Mahalanobis distance* ρ_M on $\mathbb{R}^d$ is a metric of the form:

$$\rho_M(x, x') = \sqrt{(x - x')^\top M(x - x')} = \|x - x'\|_M,$$

Mahalanobis distances

A *Mahalanobis distance* ρ_M on $\mathbb{R}^d$ is a metric of the form:

$$\rho_M(x, x') = \sqrt{(x - x')^\top M(x - x')} = \|x - x'\|_M,$$

where $M \in \mathrm{Sym}(\mathbb{R}^d)$ is a positive-definite (symmetric) matrix.

Mahalanobis distances

A *Mahalanobis distance* ρ_M on $\mathbb{R}^d$ is a metric of the form:

$$\rho_M(x, x') = \sqrt{(x - x')^\top M (x - x')} = \|x - x'\|_M,$$

where $M \in \mathrm{Sym}(\mathbb{R}^d)$ is a positive-definite (symmetric) matrix.

Geometric interpretation

- $M = A^\top A$ for some $A \in \mathbb{R}^{d \times d}$ since $M \succ 0$.

Mahalanobis distances

A *Mahalanobis distance* ρ_M on $\mathbb{R}^d$ is a metric of the form:

$$\rho_M(x, x') = \sqrt{(x - x')^\top M (x - x')} = \|x - x'\|_M,$$

where $M \in \mathrm{Sym}(\mathbb{R}^d)$ is a positive-definite (symmetric) matrix.

Geometric interpretation

- $M = A^\top A$ for some $A \in \mathbb{R}^{d \times d}$ since $M \succ 0$.
- Let $\Phi(x) = Ax$ be a new (linear) representation. Then:

$$\rho_M(x, x') = \|\Phi(x) - \Phi(x')\|_2.$$

A mathematical simplification

A user with ideal point $u \in \mathbb{R}^d$ can give two types of feedback:

A mathematical simplification

A user with ideal point $u \in \mathbb{R}^d$ can give two types of feedback:

- **Continuous responses:** measurements of the form (x, x', ψ), where:

$$\psi \equiv \psi_M(x, x'; u) = \|u - x\|_M^2 - \|u - x'\|_M^2.$$

  - Not realistic form of feedback, but mathematically easy to work with.

A mathematical simplification

A user with ideal point $u \in \mathbb{R}^d$ can give two types of feedback:

- **Continuous responses:** measurements of the form (x, x', ψ), where:

$$\psi \equiv \psi_M(x, x'; u) = \|u - x\|_M^2 - \|u - x'\|_M^2.$$

  - Not realistic form of feedback, but mathematically easy to work with.

- **Binary responses:** measurements of the form (x, x', y) where:

$$y = \mathbf{1}\{\psi < 0\}.$$

  - Later, we consider the setting where labels are binary and noisy.

A linear reparametrization (Canal et al., 2022)

Let $x, x' \in \mathbb{R}^d$ be two items. If a user has ideal point $u \in \mathbb{R}^d$, then:

$$\psi_M(x, x'; u) = \underbrace{\langle xx^\top - x'x'^\top, M\rangle}_{(1)} + \underbrace{\langle x - x', v\rangle}_{(2)}, \qquad \text{where } \underbrace{v = -2Mu}_{(3)}.$$

A linear reparametrization (Canal et al., 2022)

Let $x, x' \in \mathbb{R}^d$ be two items. If a user has ideal point $u \in \mathbb{R}^d$, then:

$$\psi_M(x, x'; u) = \underbrace{\left\langle xx^\top - x'x'^\top, M \right\rangle}_{(1)} + \underbrace{\left\langle x - x', v \right\rangle}_{(2)}, \qquad \text{where } \underbrace{v = -2Mu}_{(3)}.$$

1. $\left\langle xx^\top - x'x'^\top, M \right\rangle$ is the trace inner product on $\mathrm{Sym}(\mathbb{R}^d)$, where $\langle A, B \rangle = \mathrm{tr}(AB)$.
2. $\left\langle x - x', v \right\rangle$ is the standard inner product on $\mathbb{R}^d$.
3. The reparametrization $v = -2Mu$ is called the user's *pseudo-ideal point*.

A linear reparametrization (Canal et al., 2022)

Let $x, x' \in \mathbb{R}^d$ be two items. If a user has ideal point $u \in \mathbb{R}^d$, then:

$$\psi_M(x, x'; u) = \underbrace{\left\langle xx^\top - x'x'^\top, M \right\rangle}_{(1)} + \underbrace{\left\langle x - x', v \right\rangle}_{(2)}, \qquad \text{where } \underbrace{v = -2Mu}_{(3)}.$$

1. $\left\langle xx^\top - x'x'^\top, M \right\rangle$ is the trace inner product on $\mathrm{Sym}(\mathbb{R}^d)$, where $\langle A, B \rangle = \mathrm{tr}(AB)$.
2. $\left\langle x - x', v \right\rangle$ is the standard inner product on $\mathbb{R}^d$.
3. The reparametrization $v = -2Mu$ is called the user's *pseudo-ideal point*.

Upshot: Reparametrize (M, u) to (M, v). Then, the following map is linear:

$$(M, v) \mapsto \psi_M(x, x'; u).$$

Design matrices

Let $\{(x_{i_0}, x_{i_1})\}_{i=1}^m$ be a set of item pairs.

- Define the linear map $D : \mathrm{Sym}(\mathbb{R}^d) \oplus \mathbb{R}^d \to \mathbb{R}^m$:

$$D_i(A, w) = \left\langle x_{i_0} x_{i_0}^\top - x'_{i_1} x'^{\top}_{i_1}, A \right\rangle + \left\langle x_{i_0} - x'_{i_1}, w \right\rangle.$$

- We call D the design matrix induced by the item pairs.

Metric learning from continuous responses, single user case

Suppose a user provides us with measurements $\left\{(x_{i_0}, x_{i_1}, \psi_i)\right\}_{i=1}^{m}$, where:

$$\psi_i = \psi_M(x_{i_0}, x_{i_1}; u).$$

Metric learning from continuous responses, single user case

Suppose a user provides us with measurements $\left\{ (x_{i_0}, x_{i_1}, \psi_i) \right\}_{i=1}^{m}$, where:

$$\psi_i = \psi_M(x_{i_0}, x_{i_1}; u).$$

- We can recover (M, u) by solving the linear system of equations:

$$D_i(A, w) = \psi_i.$$

Metric learning from continuous responses, single user case

Suppose a user provides us with measurements $\left\{\left(x_{i_0}, x_{i_1}, \psi_i\right)\right\}_{i=1}^{m}$, where:

$$\psi_i = \psi_M(x_{i_0}, x_{i_1}; u).$$

- We can recover (M, u) by solving the linear system of equations:

$$D_i(A, w) = \psi_i.$$

- The pair (M, v) of the Mahalanobis matrix and pseudo-ideal point is a solution.

Metric learning from continuous responses, single user case

Suppose a user provides us with measurements $\left\{\left(x_{i_0}, x_{i_1}, \psi_i\right)\right\}_{i=1}^{m}$, where:

$$\psi_i = \psi_M(x_{i_0}, x_{i_1}; u).$$

- We can recover (M, u) by solving the linear system of equations:

$$D_i(A, w) = \psi_i.$$

- The pair (M, v) of the Mahalanobis matrix and pseudo-ideal point is a solution.
  - The ideal point can be computed from the pseudo-ideal point since $v = -2Mu$.

Metric learning from continuous responses, single user case

Suppose a user provides us with measurements $\left\{(x_{i_0}, x_{i_1}, \psi_i)\right\}_{i=1}^{m}$, where:

$$\psi_i = \psi_M(x_{i_0}, x_{i_1}; u).$$

- We can recover (M, u) by solving the linear system of equations:

$$D_i(A, w) = \psi_i.$$

- The pair (M, v) of the Mahalanobis matrix and pseudo-ideal point is a solution.
    - The ideal point can be computed from the pseudo-ideal point since $v = -2Mu$.
- To recover the metric and ideal point, $m = \frac{d(d+1)}{2} + d$ measurements is necessary.

Metric learning from continuous responses, multiple user case

Generalization to multiple users

- Suppose K users provide us with measurements (on distinct pairs of items).

Metric learning from continuous responses, multiple user case

Generalization to multiple users

- Suppose K users provide us with measurements (on distinct pairs of items).
- Recover the metric and all ideal points by solving a linear system of equations:

$$\mathbf{D}(A, w_1, \ldots, w_K) = \Psi,$$

where $A \in \mathrm{Sym}(\mathbb{R}^d)$ and each $w_k \in \mathbb{R}^d$.

Metric learning from continuous responses, multiple user case

Generalization to multiple users

- Suppose K users provide us with measurements (on distinct pairs of items).
- Recover the metric and all ideal points by solving a linear system of equations:

$$\mathbf{D}(A, w_1, \ldots, w_K) = \Psi,$$

where $A \in \mathrm{Sym}(\mathbb{R}^d)$ and each $w_k \in \mathbb{R}^d$.

Known results

- At least d measurements from each user is necessary to recover ideal points.

Metric learning from continuous responses, multiple user case

Generalization to multiple users

- Suppose K users provide us with measurements (on distinct pairs of items).
- Recover the metric and all ideal points by solving a linear system of equations:

$$\mathbf{D}(A, w_1, \ldots, w_K) = \Psi,$$

where $A \in \mathrm{Sym}(\mathbb{R}^d)$ and each $w_k \in \mathbb{R}^d$.

Known results

- At least d measurements from each user is necessary to recover ideal points.
- Recovering $(M, u_1, \ldots, u_K)$ is possible:
    - from $2d$ measurements per user if $K = \Omega(d)$

Metric learning from continuous responses, multiple user case

Generalization to multiple users

- Suppose K users provide us with measurements (on distinct pairs of items).
- Recover the metric and all ideal points by solving a linear system of equations:

$$\mathbf{D}(A, w_1, \ldots, w_K) = \Psi,$$

where $A \in \mathrm{Sym}(\mathbb{R}^d)$ and each $w_k \in \mathbb{R}^d$.

Known results

- At least d measurements from each user is necessary to recover ideal points.
- Recovering $(M, u_1, \ldots, u_K)$ is possible:
  - from $2d$ measurements per user if $K = \Omega(d)$
  - from $d + 1$ measurements per user if $K = \Omega(d^2)$.

Basic question: metric learning from lazy crowds

We ask: Suppose we can obtain very few $m \ll d$ measurements per user. Though ideal points can no longer be learned, is metric learning still possible?

Basic question: metric learning from lazy crowds

We ask: Suppose we can obtain very few $m \ll d$ measurements per user. Though ideal points can no longer be learned, is metric learning still possible?

High-level structure:

- Matrix sensing problem: learn the parameters of $M \in \mathrm{Sym}(\mathbb{R}^d)$.

Basic question: metric learning from lazy crowds

We ask: Suppose we can obtain very few $m \ll d$ measurements per user. Though ideal points can no longer be learned, is metric learning still possible?

High-level structure:

- Matrix sensing problem: learn the parameters of $M \in \mathrm{Sym}(\mathbb{R}^d)$.
- We have access to a large pool of sensors (users + items).

Basic question: metric learning from lazy crowds

We ask: Suppose we can obtain very few $m \ll d$ measurements per user. Though ideal points can no longer be learned, is metric learning still possible?

High-level structure:

- Matrix sensing problem: learn the parameters of $M \in \mathrm{Sym}(\mathbb{R}^d)$.
- We have access to a large pool of sensors (users + items).
  - Part of the measurement parameters are latent (unknown ideal points).

Basic question: metric learning from lazy crowds

We ask: Suppose we can obtain very few $m \ll d$ measurements per user. Though ideal points can no longer be learned, is metric learning still possible?

High-level structure:

- Matrix sensing problem: learn the parameters of $M \in \mathrm{Sym}(\mathbb{R}^d)$.
- We have access to a large pool of sensors (users + items).
  - Part of the measurement parameters are latent (unknown ideal points).
  - Previous work: learn latent parameters along with M.

Basic question: metric learning from lazy crowds

We ask: Suppose we can obtain very few $m \ll d$ measurements per user. Though ideal points can no longer be learned, is metric learning still possible?

High-level structure:

- Matrix sensing problem: learn the parameters of $M \in \mathrm{Sym}(\mathbb{R}^d)$.
- We have access to a large pool of sensors (users + items).
  - Part of the measurement parameters are latent (unknown ideal points).
  - Previous work: learn latent parameters along with M.
- Our regime: too few measurements per user to learn latent parameters.

An impossibility result

Setting for impossibility result

Setting.

- Let $\mathcal{X} \subset (\mathbb{R}^d, \rho_M)$ be a countable set of items.
- Let user $k \in \mathbb{N}$ have pseudo-ideal point v_k.
- We ask $m \leq d$ pairwise comparisons per user over items in $\mathcal{X}$.
  - Let $D^{(k)}$ be the design matrix for user k.

Learning latent parameters is necessary

Theorem (Impossibility result)

For (i) almost all sets $\mathcal{X}$,

Learning latent parameters is necessary

Theorem (Impossibility result)

For (i) almost all sets $\mathcal{X}$*,(ii) any set of designs* $D^{(k)}$*,*

Learning latent parameters is necessary

Theorem (Impossibility result)

For (i) almost all sets $\mathcal{X}$*,(ii) any set of designs* $D^{(k)}$*,and (iii) any* $M' \in \mathrm{Sym}(\mathbb{R}^d)$*,*

Learning latent parameters is necessary

Theorem (Impossibility result)

For (i) almost all sets $\mathcal{X}$,(ii) any set of designs $D^{(k)}$,and (iii) any $M' \in \mathrm{Sym}(\mathbb{R}^d)$,there exists $v'_k \in \mathbb{R}^d$ such that:

$$D^{(k)}(M, v_k) = D^{(k)}(M', v'_k), \qquad \forall k \in \mathbb{N}.$$

- That is, M' is consistent with observed data.

Learning latent parameters is necessary

Theorem (Impossibility result)

For (i) almost all sets $\mathcal{X}$,(ii) any set of designs $D^{(k)}$,and (iii) any $M' \in \mathrm{Sym}(\mathbb{R}^d)$,there exists $v'_k \in \mathbb{R}^d$ such that:

$$D^{(k)}(M, v_k) = D^{(k)}(M', v'_k), \qquad \forall k \in \mathbb{N}.$$

- That is, M' is consistent with observed data.
- Each user introduces enough degrees of freedom to account for all variation in data.

Learning latent parameters is necessary

Theorem (Impossibility result)

For (i) almost all sets $\mathcal{X}$*,(ii) any set of designs* $D^{(k)}$*,and (iii) any* $M' \in \mathrm{Sym}(\mathbb{R}^d)$*,there exists* $v'_k \in \mathbb{R}^d$ *such that:*

$$D^{(k)}(M, v_k) = D^{(k)}(M', v'_k), \qquad \forall k \in \mathbb{N}.$$

- That is, M' is consistent with observed data.
- Each user introduces enough degrees of freedom to account for all variation in data.
- Not only is recovery impossible, but we learn nothing at all about M.

Which sets do "almost all" item sets refer to?

Theorem (Impossibility result)

When (i) $\mathcal{X}$ has generic pairwise relations, (ii) . . . the impossibility result holds.

Which sets do "almost all" item sets refer to?

Theorem (Impossibility result)

When (i) $\mathcal{X}$ has generic pairwise relations, (ii) . . . the impossibility result holds.

- We introduce a notion of genericity, slightly stronger than *general linear position*.

Which sets do "almost all" item sets refer to?

Theorem (Impossibility result)

When (i) $\mathcal{X}$ has generic pairwise relations, (ii) . . . the impossibility result holds.

- We introduce a notion of genericity, slightly stronger than *general linear position*.
- Almost all finite sets are generic in this sense (w.r.t. Lebesgue measure on $\mathbb{R}^d$).

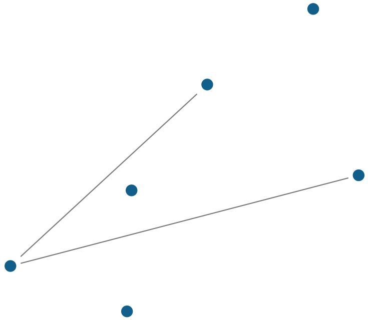
General linear position

Definition

A set $\mathcal{X} \subset \mathbb{R}^d$ is in **general linear position** *if the following is linearly independent:*

$$\{x_i - x_0 : i = 1, \ldots, n\},$$

for any distinct $x_0, x_1, \ldots, x_n \in \mathcal{X}$ and $n \leq d$.



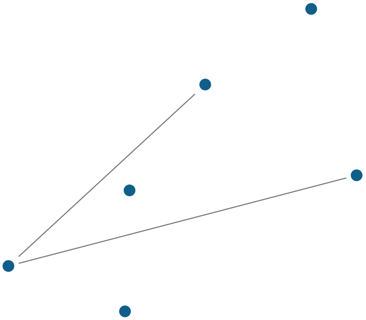
A set of points $\mathcal{X} \subset \mathbb{R}^d$.

General linear position: alternate definition

Definition

A set $\mathcal{X} \subset \mathbb{R}^d$ is in general linear position if for any star graph $G = (V \subset \mathcal{X}, E)$ with $|E| \le d$, the following is linearly independent:

$$\{x - x' : (x, x') \in E\}.$$



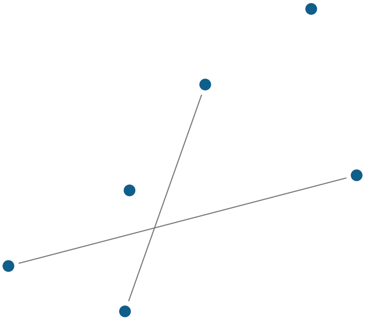
A set of points $\mathcal{X} \subset \mathbb{R}^d$.

Generic pairwise relation

Definition

A set $\mathcal{X} \subset \mathbb{R}^d$ has ***generic pairwise relations*** *if for any acyclic graph $G = (\mathcal{X}, E)$ with $|E| \leq d$, the following is linearly independent:*

$$\{x - x' : (x, x') \in E\}.$$



A set of points $\mathcal{X} \subset \mathbb{R}^d$.

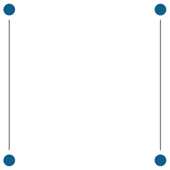
Generic pairwise relations $\Longrightarrow$ general linear position

Proof.

A star graph with at most d edges is an acyclic graph with at most d edges. $\square$

General linear position $\not\Longrightarrow$ generic pairwise relations

- ✓ General linear position—no three points are colinear.
- × These points do not have generic pairwise relations.



General takeaway I

(Not) learning from crowd data

- Weaker feedback may make data easier/cheaper to collect
    - e.g. triplet $\rightarrow$ binary feedback (with latent comparator)

General takeaway I

(Not) learning from crowd data

- Weaker feedback may make data easier/cheaper to collect
  - e.g. triplet $\rightarrow$ binary feedback (with latent comparator)
- But we may need to pay for it elsewhere
  - e.g. new fundamental limits/regimes where data carries no information

Metric learning with subspace-cluster structure

Real data often exhibit additional structure

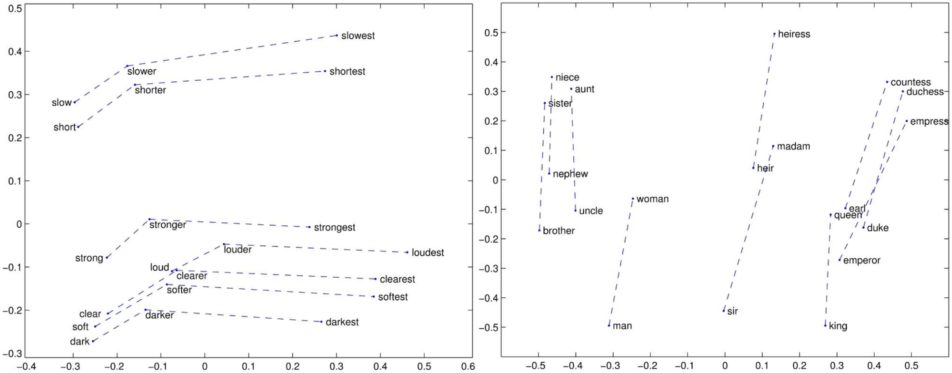


Figure 2: An example of data that approximately *does not have* generic pairwise relations (Pennington et al., 2014).

Subspace-clusterability assumption

Assumption:
There are low-dimensional subspaces of $\mathbb{R}^d$ that are 'rich' with items.

- That is, assume that $\mathcal{X}$ lies on a union of low-rank subspaces.

Subspace-clusterability assumption

Assumption:
There are low-dimensional subspaces of $\mathbb{R}^d$ that are ‘rich’ with items.

- That is, assume that $\mathcal{X}$ lies on a union of low-rank subspaces.
- e.g. $\mathcal{X}$ is *sparsely encodable*, in the sense of dictionary learning.

Divide-and-conquer approach

A natural approach:

1. Learn the metric restricted to each of the item-rich subspaces.
2. Stitch the subspace metrics together.

Subspace Mahalanobis distances

Definition

*Let $V \subset \mathbb{R}^d$ be a subspace. A metric on V is a **subspace Mahalanobis distance** if it is the subspace metric of a Mahalanobis distance ρ on $\mathbb{R}^d$,*

$$\rho\big|_V(x, x') = \rho(x, x'), \qquad \forall x, x' \in V.$$

Why can we divide?

Simple case: both items $\mathcal{X}$ and user ideal point u belong to V.

Why can we divide?

Simple case: both items $\mathcal{X}$ and user ideal point u belong to V.

- Simply reparametrize problem without the extra dimensions $V^{\perp}$.
- Learn $\rho\big|_V$ like before.

Why can we divide?

Simple case: both items $\mathcal{X}$ and user ideal point u belong to V.

- Simply reparametrize problem without the extra dimensions $V^\perp$.
- Learn $\rho\big|_V$ like before.

General case: we cannot assume the user ideal point u belongs to V.

Why can we divide?

Simple case: both items $\mathcal{X}$ and user ideal point u belong to V.

- Simply reparametrize problem without the extra dimensions $V^\perp$.
- Learn $\rho\big|_V$ like before.

General case: we cannot assume the user ideal point u belongs to V.

- It turns out for any $u \in \mathbb{R}^d$, there exists a phantom ideal point $\tilde{u}$ in V such that:

$$\psi_M(x, x'; u) = \psi_M(x, x'; \tilde{u}), \qquad \forall x, x' \in V.$$

Why can we divide?

Simple case: both items $\mathcal{X}$ and user ideal point u belong to V.

- Simply reparametrize problem without the extra dimensions $V^{\perp}$.
- Learn $\rho\big|_V$ like before.

General case: we cannot assume the user ideal point u belongs to V.

- It turns out for any $u \in \mathbb{R}^d$, there exists a phantom ideal point $\tilde{u}$ in V such that:

$$\psi_M(x, x'; u) = \psi_M(x, x'; \tilde{u}), \qquad \forall x, x' \in V.$$

- We can no longer recover u, but we can learn $\rho\big|_V$.

Why can we recombine?

After dividing, we end up with a collection of subspace metric:

$$\rho\big|_{V_1}, \ldots, \rho\big|_{V_n}.$$

Why can we recombine?

After dividing, we end up with a collection of subspace metric:

$$\rho\big|_{V_1}, \ldots, \rho\big|_{V_n}.$$

Result: As long as the subspaces $V_1, \ldots, V_n$ *quadratically span* $\mathbb{R}^d$, there is a unique Mahalanobis distance on $\mathbb{R}^d$ generating the joint subspace metrics.

Geometric proof

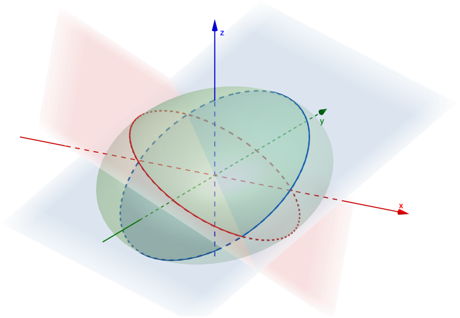


Figure 3: Unit spheres of Mahalanobis distances are ellipsoids in $\mathbb{R}^d$.

Geometric proof

For Mahalanobis distances:

- Metric learning is equivalent to recovering its unit ellipsoid $\mathcal{E}$.

Geometric proof

For Mahalanobis distances:

- Metric learning is equivalent to recovering its unit ellipsoid $\mathcal{E}$.
- Learning the subspace metric on V correspond to recovering the slice $V \cap \mathcal{E}$.

Geometric proof

For Mahalanobis distances:

- Metric learning is equivalent to recovering its unit ellipsoid $\mathcal{E}$.
- Learning the subspace metric on V correspond to recovering the slice $V \cap \mathcal{E}$.

Fact from geometry:
We can reconstruct an ellipsoid given enough low-dimensional slices.

Quadratic spanning

Definition

The subspaces $V_1, \ldots, V_n \subset \mathbb{R}^d$ ***quadratically span*** $\mathbb{R}^d$ *if the (linear) span satisfies:*

$$\mathrm{Sym}(\mathbb{R}^d) = \mathrm{span}\left(\left\{xx^\top : x \in V_1 \cup \cdots \cup V_n\right\}\right).$$

Metric learning from lazy crowds (simple math setting)

We asked: Suppose we can obtain very few $m \ll d$ measurements per user. Though ideal points can no longer be learned, is metric learning still possible?

Metric learning from lazy crowds (simple math setting)

We asked: Suppose we can obtain very few $m \ll d$ measurements per user. Though ideal points can no longer be learned, is metric learning still possible?

Answer (continuous response model):

- In general, this is not possible.

Metric learning from lazy crowds (simple math setting)

We asked: Suppose we can obtain very few $m \ll d$ measurements per user. Though ideal points can no longer be learned, is metric learning still possible?

Answer (continuous response model):

- In general, this is not possible.
- If $\mathcal{X}$ is a union of r-dimensional subspaces ($r \ll d$), it is possible with:

number of users	d^2/r
measurements per user	$2r$

General takeaway II

Learning from crowd data

- Fundamental limit overcome using additional structural assumptions
  - e.g. generic pairwise relations $\rightarrow$ subspace-cluster structure

General takeaway II

Learning from crowd data

- Fundamental limit overcome using additional structural assumptions
  - e.g. generic pairwise relations $\rightarrow$ subspace-cluster structure
- These structural assumptions could be (approximately) realistic
  - we could even enforce the structure upsteam
  - e.g. generate representations via dictionary learning

Goals of the rest of the talk

Up to now:

- Fundamental limits of weak and per-user-budgeted crowdsourced data
- Paying for weak feedback if there is additional structure

Goals of the rest of the talk

Up to now:

- Fundamental limits of weak and per-user-budgeted crowdsourced data
- Paying for weak feedback if there is additional structure

Rest of the talk:

- High-level description of statistical/learning-theoretic techniques
- A commonly used model for analyzing preference feedback
- A fundamental open question: crowdsourced sensing with latent parameters

Metric learning from non-idealized data

Divide-and-conquer for idealized data

Divide step:
For each subspace $V_1, \ldots, V_n$, solve a system of linear equations:

$$\mathbf{D}_i(\hat{Q}_i, w_1, \ldots, w_K) = \Psi_i.$$

Recombine step:
Define $\Pi(M) = (Q_1, \ldots, Q_n)$ to be the linear map:

$$\Pi : \text{parameters of Mahalanobis distances} \mapsto \text{parameters of subspace metrics}.$$

Solve a system of linear equations:

$$\hat{M} = \Pi(\hat{Q}_1, \ldots, \hat{Q}_n).$$

From linear systems to regression

Question: What happens in the non-idealized setting?

From linear systems to regression

Question: What happens in the non-idealized setting?

- Feedback is inexact/binary/noisy.
- The set of items is only approximately subspace clusterable.

From linear systems to regression

Question: What happens in the non-idealized setting?

- Feedback is inexact/binary/noisy.
- The set of items is only approximately subspace clusterable.

Divide step:
Prior work shows shows metric learning from non-idealized feedback.

- If we get binary responses, solve a binary regression problem instead.

From linear systems to regression

Question: What happens in the non-idealized setting?

- Feedback is inexact/binary/noisy.
- The set of items is only approximately subspace clusterable.

Divide step:
Prior work shows shows metric learning from non-idealized feedback.

- If we get binary responses, solve a binary regression problem instead.

Recombine step:
We need to show that we can recombine estimated subspace metrics.

- **Algorithm:** perform linear regression instead, and project onto the PSD cone.

Setting for recombination recovery guarantee

Setting:

- Let $\mathcal{V}_n = \{V_1, \ldots, V_n\}$ be a collection of subspaces of $\mathbb{R}^d$.

Setting for recombination recovery guarantee

Setting:

- Let $\mathcal{V}_n = \{V_1, \ldots, V_n\}$ be a collection of subspaces of $\mathbb{R}^d$.
- Let $Q_1, \ldots, Q_n$ be the true parameters of the subspace metrics.

Setting for recombination recovery guarantee

Setting:

- Let $\mathcal{V}_n = \{V_1, \ldots, V_n\}$ be a collection of subspaces of $\mathbb{R}^d$.
- Let $Q_1, \ldots, Q_n$ be the true parameters of the subspace metrics.
- Let $\hat{Q}_1, \ldots, \hat{Q}_n$ be independent estimators of the subspace metrics.

Setting for recombination recovery guarantee

Setting:

- Let $\mathcal{V}_n = \{V_1, \ldots, V_n\}$ be a collection of subspaces of $\mathbb{R}^d$.
- Let $Q_1, \ldots, Q_n$ be the true parameters of the subspace metrics.
- Let $\hat{Q}_1, \ldots, \hat{Q}_n$ be independent estimators of the subspace metrics.
- Let $\hat{M}$ be the projected ordinary least squares solution (on the PSD cone):

$$\hat{M}_{\mathrm{OLS}} = \operatorname*{arg\,min}_{A \in \mathrm{Sym}(\mathbb{R}^d)} \sum_{i=1}^{n} \|\hat{Q}_i - \Pi_{V_i}(A)\|^2$$

$$\hat{M} = \operatorname*{arg\,min}_{A \succeq 0} \|\hat{M}_{\mathrm{OLS}} - A\|_F^2.$$

Recombination recovery guarantee

Assumptions:

- The estimators have low-bias: $\| \mathbb{E}[\hat{Q}_i] - Q_i\| \leq \gamma$.
- The estimators have bounded spread: $\|\hat{Q}_i - \mathbb{E}[\hat{Q}_i]\| \leq \varepsilon$.

Recombination recovery guarantee

Assumptions:

- The estimators have low-bias: $\|\mathbb{E}[\hat{Q}_i] - Q_i\| \leq \gamma$.
- The estimators have bounded spread: $\|\hat{Q}_i - \mathbb{E}[\hat{Q}_i]\| \leq \varepsilon$.

Theorem

There is a constant $c > 0$ such that for any $p \in (0, 1]$, with probability at least $1 - p$,

$$\|\hat{M} - M\|_F \leq c \cdot \frac{1}{\sigma(\mathcal{V}_n)} \cdot \left(\gamma\sqrt{n} + \varepsilon d \sqrt{\log \frac{2d}{p}} \right),$$

where $\sigma(\mathcal{V})$ quantifies the 'quadratic spread' of subspaces $V_1, \ldots, V_n$ in $\mathrm{Sym}(\mathbb{R}^d)$.

Proof sketch

For simplicity, we just show bound for $\|\hat{M}_{\text{OLS}} - M\|_F$.

1. Recall the linear map $\Pi(M) = (Q_1, \ldots, Q_n)$.

Proof sketch

For simplicity, we just show bound for $\|\hat{M}_{\text{OLS}} - M\|_F$.

1. Recall the linear map $\Pi(M) = (Q_1, \ldots, Q_n)$.
2. The OLS solution is computed by the Moore-Penrose pseudoinverse:

$$\hat{M}_{\text{OLS}} = \Pi^+(\hat{Q}_1, \ldots, \hat{Q}_n).$$

Proof sketch

For simplicity, we just show bound for $\|\hat{M}_{\text{OLS}} - M\|_F$.

1. Recall the linear map $\Pi(M) = (Q_1, \ldots, Q_n)$.
2. The OLS solution is computed by the Moore-Penrose pseudoinverse:
$$\hat{M}_{\text{OLS}} = \Pi^+(\hat{Q}_1, \ldots, \hat{Q}_n).$$
3. Since Π^+ is linear, we get the bound:
$$\|\hat{M}_{\text{OLS}} - M\|_F^2 = \|\Pi^+(\hat{Q}_1 - Q, \ldots, \hat{Q}_n - Q_n)\|_F^2 \leq \sigma_{\max}^2(\Pi^+) \sum_{i=1}^{n} \|\hat{Q}_1 - Q_i\|^2.$$

Proof sketch

For simplicity, we just show bound for $\|\hat{M}_{\mathrm{OLS}} - M\|_F$.

1. Recall the linear map $\Pi(M) = (Q_1, \ldots, Q_n)$.
2. The OLS solution is computed by the Moore-Penrose pseudoinverse:

$$\hat{M}_{\mathrm{OLS}} = \Pi^+(\hat{Q}_1, \ldots, \hat{Q}_n).$$

3. Since Π^+ is linear, we get the bound:

$$\|\hat{M}_{\mathrm{OLS}} - M\|_F^2 = \|\Pi^+(\hat{Q}_1 - Q, \ldots, \hat{Q}_n - Q_n)\|_F^2 \leq \sigma_{\max}^2(\Pi^+) \sum_{i=1}^{n} \|\hat{Q}_1 - Q_i\|^2.$$

4. A more fine-grained bound by decomposition: $\hat{Q} - Q = \underbrace{\hat{Q} - \mathbb{E}[\hat{Q}]}_{\text{mean-zero r.v.}} + \underbrace{\mathbb{E}[\hat{Q}] - Q}_{\text{bias}}.$

Proof sketch

For simplicity, we just show bound for $\|\hat{M}_{\text{OLS}} - M\|_F$.

1. Recall the linear map $\Pi(M) = (Q_1, \ldots, Q_n)$.
2. The OLS solution is computed by the Moore-Penrose pseudoinverse:

$$\hat{M}_{\text{OLS}} = \Pi^{+}(\hat{Q}_1, \ldots, \hat{Q}_n).$$

3. Since Π^{+} is linear, we get the bound:

$$\|\hat{M}_{\text{OLS}} - M\|_F^2 = \|\Pi^{+}(\hat{Q}_1 - Q, \ldots, \hat{Q}_n - Q_n)\|_F^2 \leq \sigma_{\max}^2(\Pi^{+}) \sum_{i=1}^{n} \|\hat{Q}_1 - Q_i\|^2.$$

4. A more fine-grained bound by decomposition: $\hat{Q} - Q = \underbrace{\hat{Q} - \mathbb{E}[\hat{Q}]}_{\text{mean-zero r.v.}} + \underbrace{\mathbb{E}[\hat{Q}] - Q}_{\text{bias}}$.
   - For independent mean-zero error terms, can apply Chernoff-style concentration.

Interpretation of the bound

Key quantities: n = number of subspaces; γ, ε = subspace recovery bias/accuracy

$$\|\hat{M} - M\|_F \leq c \cdot \frac{1}{\sigma(\mathcal{V})} \cdot \left(\gamma\sqrt{n} + \varepsilon d \sqrt{\log \frac{2d}{p}} \right)$$

Interpretation of the bound

Key quantities: $n =$ number of subspaces; $\gamma, \varepsilon =$ subspace recovery bias/accuracy

$$\|\hat{M} - M\|_F \leq c \cdot \frac{1}{\sigma(\mathcal{V})} \cdot \left(\gamma\sqrt{n} + \varepsilon d \sqrt{\log \frac{2d}{p}} \right)$$

- $\sigma(\mathcal{V}_n)$ grows with the number of subspaces,

$$\sigma(\mathcal{V}_n) = \Omega(\sqrt{n}) \text{ is possible.}$$

Interpretation of the bound

Key quantities: n = number of subspaces; γ, ε = subspace recovery bias/accuracy

$$\|\hat{M} - M\|_F \leq c \cdot \frac{1}{\sigma(\mathcal{V})} \cdot \left(\gamma\sqrt{n} + \varepsilon d \sqrt{\log \frac{2d}{p}} \right)$$

- $\sigma(\mathcal{V}_n)$ grows with the number of subspaces,

$$\sigma(\mathcal{V}_n) = \Omega(\sqrt{n}) \text{ is possible.}$$

As $n \to \infty$, the dominating term is possibly the bias term γ.

- e.g. if the estimators $\hat{Q}$ have a systematic constant biases $\gamma > 0$.

A noisy feedback model with recovery guarantee

Probabilistic model

Generalized linear model:

- Continuous response: (x, x', ψ)

$$\psi \equiv \psi_M(x, x'; u) = D_{x,x'}(M, v).$$

Probabilistic model

Generalized linear model:

- Continuous response: (x, x', ψ)

$$\psi \equiv \psi_M(x, x'; u) = D_{x,x'}(M, v).$$

- Noisy response: (x, x', y)

$$\Pr\left[Y = y \,\middle|\, M, x, x', u\right] = f\big(y \cdot D_{x,x'}(M, v)\big),$$

where f is a (non-linear) *link function*.

Probabilistic model

Generalized linear model:

- Continuous response: (x, x', ψ)

$$\psi \equiv \psi_M(x, x'; u) = D_{x,x'}(M, v).$$

- Noisy response: (x, x', y)

$$\Pr\left[Y = y \,\middle|\, M, x, x', u\right] = f\big(y \cdot D_{x,x'}(M, v)\big),$$

where f is a (non-linear) *link function.*

  - The link function is the first (and only) instance of a non-linearity in this work.

Probabilistic model

Generalized linear model:

- Continuous response: (x, x', ψ)

$$\psi \equiv \psi_M(x, x'; u) = D_{x,x'}(M, v).$$

- Noisy response: (x, x', y)

$$\Pr\left[Y = y \,\middle|\, M, x, x', u\right] = f\big(y \cdot D_{x,x'}(M, v)\big),$$

  where f is a (non-linear) *link function.*
  - The link function is the first (and only) instance of a non-linearity in this work.
  - When $f(z) = \frac{1}{1+\exp(-z)}$ is the sigmoid function, this leads to a logistic regression.

Setting for subspace metric recovery

Setting:

- Assume that user provide measurements (x, x', Y) where $Y \in \{-1, +1\}$,

$$\Pr\left[Y = y\right] = f\big(-y \cdot D_{x,x'}(M, v)\big),$$

where f is the sigmoid link function.

Setting for subspace metric recovery

Setting:

- Assume that user provide measurements (x, x', Y) where $Y \in \{-1, +1\}$,

$$\Pr\left[Y = y\right] = f\big(- y \cdot D_{x,x'}(M, v)\big),$$

  where f is the sigmoid link function.

- We can perform maximum likelihood estimation:

$$(\hat{M}, \hat{v}_1, \ldots, \hat{v}_k) \leftarrow \operatorname*{arg\,max}_{(A, w_1, \ldots, w_K)} \sum_{k} \sum_{(x,x',Y)} \log f\big(- Y \cdot D_{x,x'}(M, v_k)\big).$$

Setting for subspace metric recovery

Setting:

- Assume that user provide measurements (x, x', Y) where $Y \in \{-1, +1\}$,

$$\Pr\left[Y = y\right] = f\big(- y \cdot D_{x,x'}(M, v)\big),$$

  where f is the sigmoid link function.

- We can perform maximum likelihood estimation:

$$(\hat{M}, \hat{v}_1, \ldots, \hat{v}_k) \leftarrow \underset{(A, w_1, \ldots, w_K)}{\arg\max} \sum_k \sum_{(x,x',Y)} \log f\big(- Y \cdot D_{x,x'}(M, v_k)\big).$$

- Assume $\|M\|_\infty \leq 1$ and items and ideal points are contained in unit Euclidean ball.

Analysis via generalization

Theorem (Metric recovery, adapted from Canal et al. (2022))

Let $\mathcal{X}$ quadratically span $\mathbb{R}^d$. There exists designs $D^{(k)}$ asking for m responses from each of K users such that from that data, the maximum likelihood estimator $\hat{M}$ satisfies w.h.p.:

$$\|\hat{M} - M\|_F^2 = \mathcal{O}\left(\sqrt{\frac{d^2 + dK}{mK}}\right).$$

Analysis via generalization

Theorem (Metric recovery, adapted from Canal et al. (2022))

Let $\mathcal{X}$ quadratically span $\mathbb{R}^d$. There exists designs $D^{(k)}$ asking for m responses from each of K users such that from that data, the maximum likelihood estimator $\hat{M}$ satisfies w.h.p.:

$$\|\hat{M} - M\|_F^2 = \mathcal{O}\left(\sqrt{\frac{d^2 + dK}{mK}}\right).$$

- Proof uses standard techniques from generalization theory.
- The $d^2 + dK$ term comes from a metric entropy bound on:

$$\{(A, u_1, \dots, u_K) : \|A\|_\infty \leq 1 \text{ and } \|u_k\| \leq 1, \forall k\}.$$

- When $K \gg d^2$, the dominating term is $\sqrt{d/m}$.

Open question

Weakness of analysis, weakness of naive ERM, or fundamental limit?

- The generalization approach actually shows:

$$\|\hat{M} - M\|_F^2 + \sum_{k=1}^{K} \|\hat{v}_k - v_k\|^2 = \mathcal{O}\left(\sqrt{\frac{d^2 + dK}{mK}}\right).$$

- But, we only care about learning the parameters of M.

Open question

Weakness of analysis, weakness of naive ERM, or fundamental limit?

- The generalization approach actually shows:

$$\|\hat{M} - M\|_F^2 + \sum_{k=1}^{K} \|\hat{v}_k - v_k\|^2 = \mathcal{O}\left(\sqrt{\frac{d^2 + dK}{mK}}\right).$$

- But, we only care about learning the parameters of M.
- This analysis does not seem to allow us to decouple estimating $\hat{M}$ and $\hat{v}_k$.
- Is the analysis loose? Is there a better algorithm? Is there a fundamental limit?

Implication for metric learning

Suppose K users provide m measurements on rank-r subspaces.

Subspace metric error:

$$\gamma + \varepsilon \leq \mathcal{O}\left(\sqrt{\frac{r^2 + rK}{mK}}\right).$$

Metric error after recombination:

$$\|\hat{M} - M\|_F \leq c \cdot \frac{1}{\sigma(\mathcal{V})} \cdot \left(\gamma\sqrt{n} + \varepsilon d \sqrt{\log \frac{2d}{p}}\right)$$

When $K \gg d$, then there are settings with: $\|\hat{M} - M\|_F = \mathcal{O}\left(\sqrt{\frac{r}{m}}\right)$.

Additional open problems

Further questions

Other structure:

- Low rank metrics; non-linear representations/kernel extension
- Learning with approximate subspace clusters
- Learning with structured user sets

Inducing structure:

- What are good representations for human/crowdsourced labeling?

Statistics:

- Other noise/preference models (e.g. Bradley-Terry model)
- Semi-parametric estimation
- Robust recovery

Acknowledgments

Collaborators

Zhi Wang
UC San Diego

Ramya Korlakai Vinayak
UW–Madison

Thank you!

See *https://geelon.github.io/* for preprint.

References

Gregory Canal, Blake Mason, Ramya Korlakai Vinayak, and Robert Nowak. One for all: Simultaneous metric and preference learning over multiple users. *Advances in Neural Information Processing Systems*, 35:4943–4956, 2022.

Clyde H Coombs. Psychological scaling without a unit of measurement. *Psychological review*, 57(3):145, 1950.

Ian J Goodfellow, Jonathon Shlens, and Christian Szegedy. Explaining and harnessing adversarial examples. *arXiv preprint arXiv:1412.6572*, 2014.

Blake Mason, Lalit Jain, and Robert Nowak. Learning low-dimensional metrics. *Advances in neural information processing systems*, 30, 2017.

Jeffrey Pennington, Richard Socher, and Christopher D Manning. Glove: Global vectors for word representation. In *Proceedings of the 2014 conference on empirical methods in natural language processing (EMNLP)*, pages 1532–1543, 2014.

Matthew Schultz and Thorsten Joachims. Learning a distance metric from relative comparisons. *Advances in neural information processing systems*, 16, 2003.

Nakul Verma and Kristin Branson. Sample complexity of learning mahalanobis distance metrics. *Advances in neural information processing systems*, 28, 2015.

Zhi Wang, Geelon So, and Ramya Korlakai Vinayak. Metric learning from limited pairwise preference comparisons, 2024.

Austin Xu and Mark Davenport. Simultaneous preference and metric learning from paired comparisons. *Advances in Neural Information Processing Systems*, 33:454–465, 2020.