# Learning mixed equilibria

#### Fudenberg and Kreps (1993)

Geelon So, agso@ucsd.edu Game theory reading group — February 13, 2023

### Learning in strategic settings

#### Question

If players play a game repeatedly, how do their beliefs and behaviors evolve over time?

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- Let  $u = (u^{(A)}, u^{(B)})$  be their utilities

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Alice can also have a *subjective belief* about how Bob learns and behaves.

# Fictitious play

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> Alice observes Bob's actual play *s*, and updates the counter for *s*:

$$N_s \leftarrow N_s + 1.$$

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Remark. Alice believes that Bob is non-adaptive, and therefore plays myopically.

► This belief justifies maximizing immediate utility.

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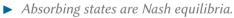
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▶ If the above equation holds, then  $s_t$  is a Nash equilibrium.

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**Remark.** The convergence of empirical frequencies is not guaranteed.

- Shapley (1964) gives the first nontrivial example; rock-paper-scissors.
- Convergence is guaranteed for zero-sum games (Robinson, 1951) and two-by-two games (Miyasawa, 1961).

Extensions of fictitious play

## Myopic behavior

#### Definition

Let  $\mu$  be the assessment rule for Alice. Her behavior rule  $\phi$  is myopic relative to  $\mu$  if:

$$\phi(\zeta_t) = \underset{\sigma \in \Sigma^{(A)}}{\arg \max} \ u^{(A)}(\sigma, \mu(\zeta_t)).$$

That is, she maximizes her immediate payoff with respect to her assessment  $\mu(\zeta_t)$  at time t.

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- ► The behavior rule of fictitious play is myopic.
- ▶ We can weaken myopia to an asymptotic notion.

## Asymptotic myopia

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At each time step, let the optimal expected utility with respect to the assessment  $\mu(\zeta_t)$  be:

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 $\blacktriangleright \phi$  is strongly asymptotically myopic if:

$$\lim_{t\to\infty}\left(u_t^*-\min_{s\in\operatorname{supp}(\phi(\zeta_t))}u^{(A)}(s,\mu(\zeta_t))\right)=0.$$

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**Justifications.** In (1), players play very infrequently, and so the future is highly discounted. In (2) and (3), the players may believe that their actions will have little influence, so that the other population is essentially non-adaptive.

#### Definition

Alice's assessment rule  $\mu$  is **adaptive** if for every  $\varepsilon > 0$  and t, if Bob did not play a set of pure strategies S' during times t to t', then Alice's assessment of this strategy set is small:

 $\phi_{t'}^{(A)}(S') < \varepsilon,$ 

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- ► That is, if Bob stops playing some strategy, then Alice's assessment that that strategy will be played converges to zero.
- ► The empirical assessment from fictitious play is adaptive.
- ► This is a fairly weak condition.

# Absorbing states are Nash equilibria

#### Proposition

Let Alice and Bob be strongly asymptotically myopic players with adaptive assessment rules. If  $s_1, s_2, ...$  is their sequence of plays and  $s_t = s_*$  for all  $t \ge T$ , then  $s_*$  is a Nash equilibrium of the game.

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3. Because Alice is strongly asymptotically myopic, given enough time, she plays arbitrarily near-optimal responses to  $\phi_t^{(A)}$ , which must be nearly as good as  $\hat{s}$ . But then,  $\mathbf{s}_t$  does not remain on  $\mathbf{s}_*$  for all time. Convergence to mixed strategies

#### Definition

Let  $\mathbf{s}_1, \mathbf{s}_2, \ldots$  be the sequence of plays. Their sequence of **empirical frequencies**  $\overline{\sigma}_t \in \Sigma^{(A)} \times \Sigma^{(B)}$ , where  $\overline{\sigma}_t$  is the empirical distribution over plays by time t.

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- ▶ Note that  $\overline{\sigma}_t$  is not necessarily a product distribution.
- ▶ If the marginal frequencies have limits,  $\lim_{t\to\infty} \overline{\sigma}_t^{(A)} = \overline{\sigma}_*^{(A)}$  and  $\lim_{t\to\infty} \overline{\sigma}_t^{(B)} = \overline{\sigma}_*^{(B)}$  exist,

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$$\sigma_*^{\otimes} = \overline{\sigma}_*^{(A)} \otimes \overline{\sigma}_*^{(B)}.$$

# Convergence implies Nash equilibrium?

**Question.** What sort of learners do Alice and Bob need to be so that we can obtain the guarantee that if  $\sigma_*^{\otimes}$  exists, then it is a mixed Nash equilibrium?

# Asymptotically empirical assessments

### Definition

Alice's assessment rule  $\mu$  is **asymptotically empirical** if:

$$\lim_{t\to\infty} \left\| \mu_t - \overline{\sigma}_t^{(B)} \right\| = 0.$$

That is, her assessment converges to Bob's empirical frequencies of play.

# Convergence implies mixed Nash equilibrium

### Proposition

Let Alice and Bob be strongly asymptotically myopic players with asymptotically empirical assessment rules. If  $\overline{\sigma}_*^{\otimes}$  exists, then it is a Nash equilibrium.

### Proof.

Analogous to proof for convergence to pure Nash equilibria.

Convergence of the empirical marginal frequencies does not imply convergence of behavior to a Nash equilibrium.

 Often, in fictitious play, players jump from pure strategies to another, typically in cycles of increasing lengths (e.g. rock-paper-scissors).



























### References

Drew Fudenberg and David M Kreps. Learning mixed equilibria. *Games and economic behavior*, 5(3): 320–367, 1993.