Nearest neighbor for realizable online classification

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EnCORE Student Social — Mar 20, 2023

Weather forecasting problem

THE WEATHER CHANNEL'S TASK

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Each day:

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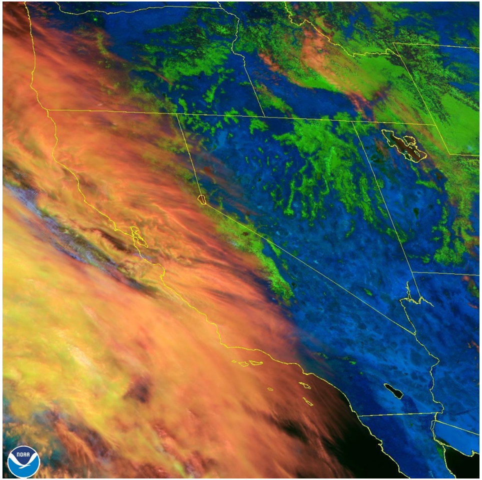

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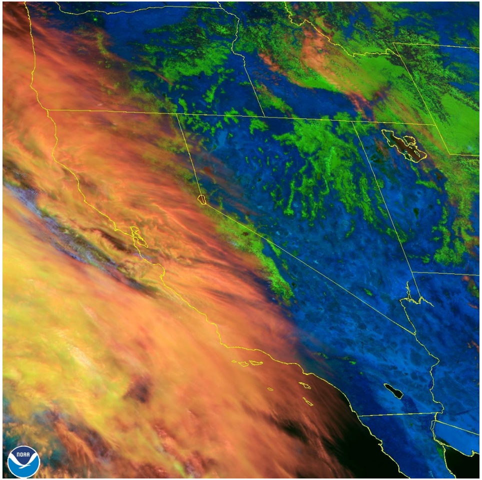

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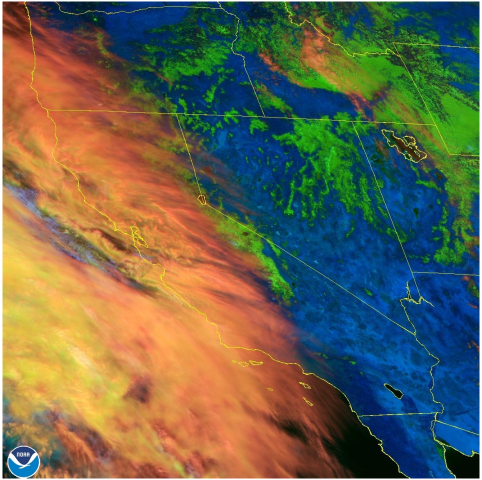

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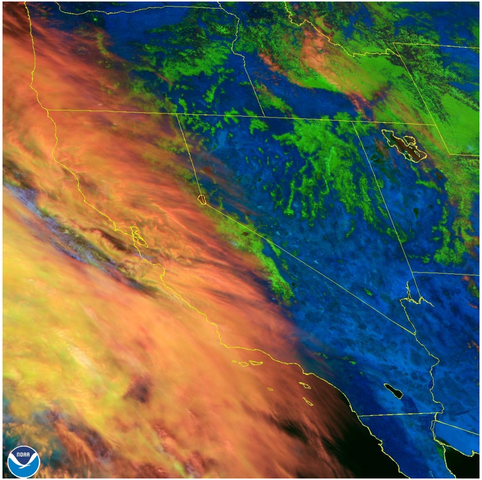

Nearest neighbor for weather prediction

NEAREST NEIGHBOR ALGORITHM

- remember all past conditions + weather outcomes

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- predict weather according to the most similar conditions in memory

The nearest neighbor rule

SETTING

Let $(\mathcal{X}, \rho)$ be a metric space.

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$$\mathrm{NN}(x) = \arg\min_{\tau} \rho(x, x_\tau)$$

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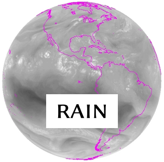
- predict using corresponding label

$$\hat{y}(x) = y_{\mathrm{NN}(x)}$$

Nearest neighbor for weather prediction



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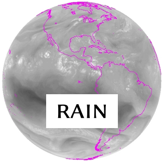

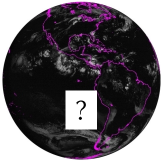

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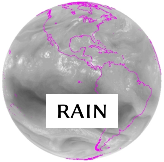

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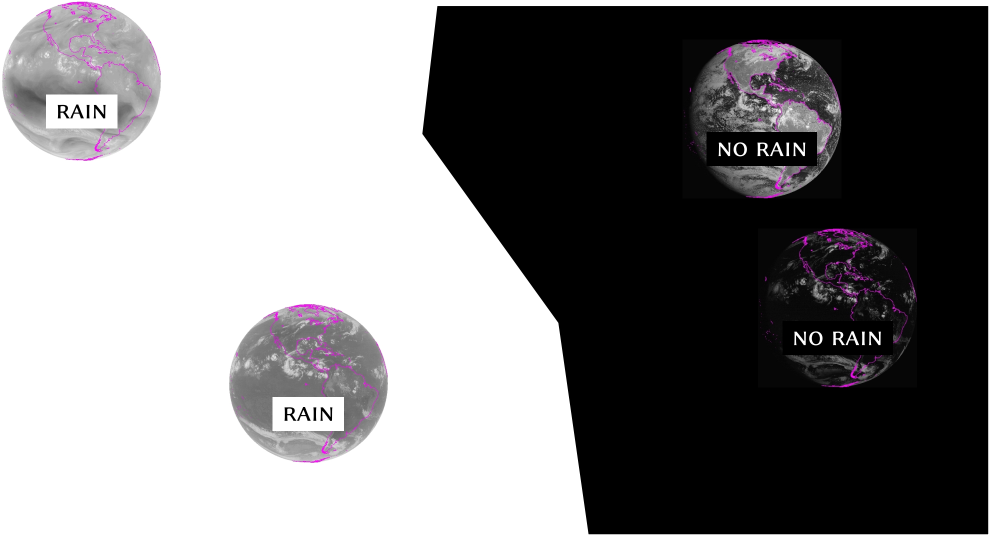

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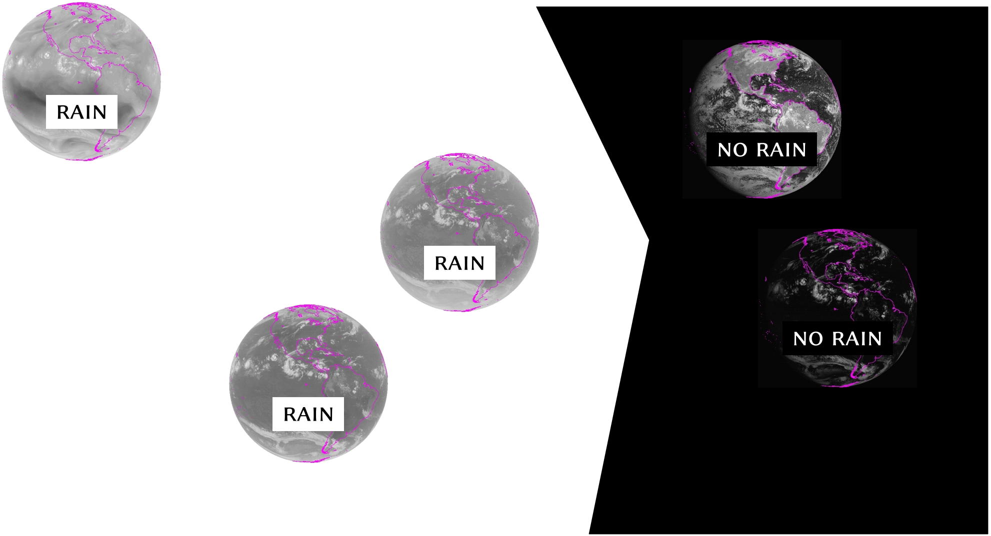

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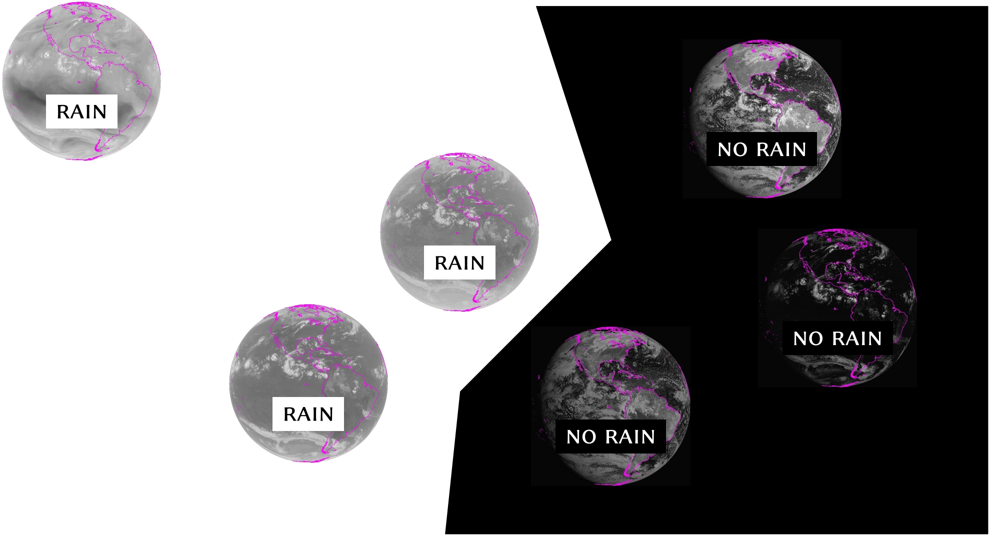

Behavior of online nearest neighbor

QUESTION

When is the *nearest neighbor rule* a reasonable online prediction strategy?

Online learning setting

ONLINE LEARNING LOOP

For $t = 1, 2, \ldots$

- receive instance x_t

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- incur loss $\ell(x_t, y_t, \hat{y}_t)$

Online learning setting

REALIZABILITY ASSUMPTION

The true labels are generated by some underlying function $f : \mathcal{X} \to \mathcal{Y}$,

$$y_t = f(x_t).$$

Online learning setting

GOAL

Make fewer and fewer mistakes over time.

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Make fewer and fewer mistakes over time. Formally:

$$\underbrace{\mathrm{er}_T := \frac{1}{T}\sum_{t=1}^{T} \ell\big(x_t, y_t, \hat{y}_t\big) \to 0}_{\text{achieve vanishing error rate}} .$$

Connection to regret

In the usual goal in the online learning setting is to achieve sublinear regret:

$$\text{regret}_T := \sum_{t=1}^{T} \ell\big(x_t, y_t, \hat{y}_t\big) - \inf_{h \in \mathcal{H}} \sum_{t=1}^{T} \ell\big(x_t, y_t, h(x_t)\big).$$

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- In the realizable setting, if $\mathcal{H}$ is non-parametric (e.g. all nearest neighbor classifiers), no mistakes are made by any optimal $h \in \mathcal{H}$ on $(x_1, y_1), \ldots, (x_T, y_T)$.

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- Thus, sublinear regret is equivalent to vanishing error rate.

Difficulty of realizable online learning

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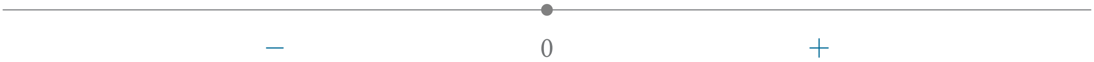
- The sequence of instances x_t do not come i.i.d. from some distribution.
- In the worst-case, each x_t is selected so that learner makes a mistake each time.

Negative example: learning the sign function

GOAL

Learn the sign function $f(x) := \begin{cases} + & x \geq 0 \\ - & x < 0 \end{cases}$



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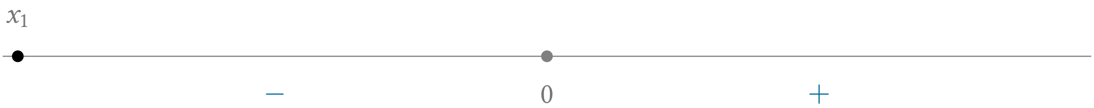

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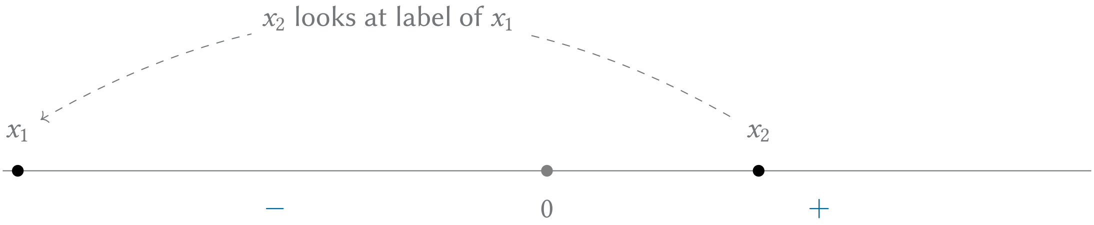

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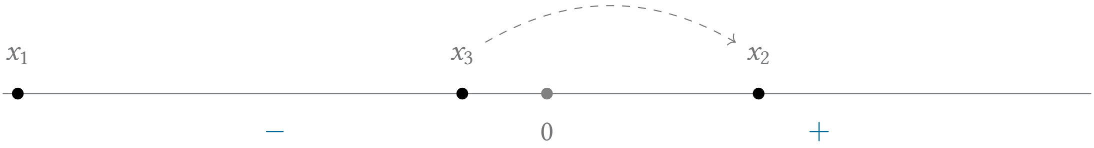

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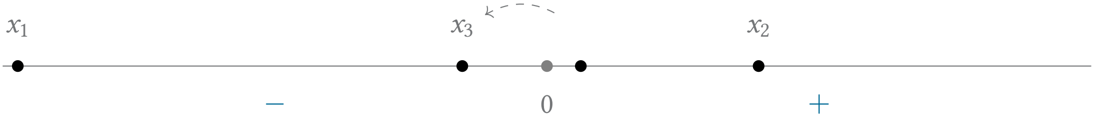

EXAMPLE. A worst-case sequence where the nearest neighbor rule errs every time.

- The sequence alternate signs and the nearest neighbor of x_{t+1} is x_t out of $x_1, \ldots, x_t$.
- Mistake rate fails to go to zero despite the mistake set shrinking exponentially fast.

Generalized negative result

SETTING

Let $(\mathcal{X}, \rho)$ be a totally bounded metric space and $f : \mathcal{X} \to \{-, +\}$.

Proposition (Non-convergence in the worst-case)

There is a sequence of instances $(x_t)_t$ on which the nearest neighbor error rate is bounded away from zero if and only if there is no positive separation between classes:

$$\inf_{f(x) \neq f(x')} \rho(x, x') = 0.$$

- **Proof idea:** can always find arbitrarily close pairs (x, x') with opposite signs
  - can select sequence so that x_{2t} is closest to x_{2t-1}, which has the opposite sign

Implications of negative result

The worst-case adversary is too powerful—learning may not be possible in this setting.

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This work

RESEARCH QUESTION

Under what *general conditions* is realizable online learning possible?

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Under what *general conditions* is realizable online learning possible?

- How much do we need to relax the worst-case adversary?

Smoothed analysis of algorithms

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- Non-worst-case analysis:
  - Introduce a (probability) measure over problem instances.
  - Show that almost all problems are easy (the hard instances have measure zero).
    - Or, problems are easy with high probability/on average.

Smoothed adversary for online learning

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The smoothed online setting is also studied by Rakhlin et al. (2011); Haghtalab et al. (2020).

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GAUSSIAN-SMOOTHED ADVERSARY:

- adversary selects $\overline{x}$
- test instance x is a perturbed version $\overline{x} + \xi$ where $\xi \sim \mathcal{N}(0, \sigma^2 I)$, so:

$$\mu = \mathcal{N}(\bar{x}, \sigma^2 I).$$

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σ-SMOOTHED ADVERSARY:

- let ν be an underlying distribution over $\mathcal{X}$
- the adversary can select any distribution μ satisfying:

$$\mu(A) \leq \frac{1}{\sigma} \cdot \nu(A),$$

for all $A \subset \mathcal{X}$ measurable.

Dominated adversary

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Definition (Dominated adversary)

The measure ν uniformly dominates a family $\mathcal{M}$ of probability distributions on $\mathcal{X}$ if for all $\varepsilon > 0$ there exists $\delta > 0$ such that:

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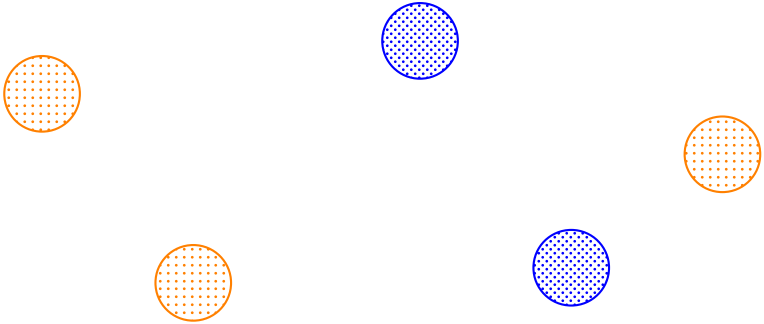
for all $A \subset \mathcal{X}$ measurable and distribution $\mu \in \mathcal{M}$. We say that adversary is ν-dominated if at all times t it selects μ_t from a family of distributions uniformly dominated by ν.

Example: learning labels for well-separated clusters

SETTING

Suppose that the instance space $\mathcal{X}$ consists of countably many well-separated clusters



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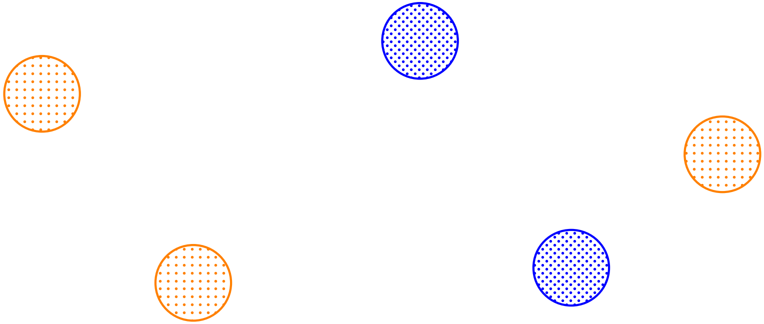

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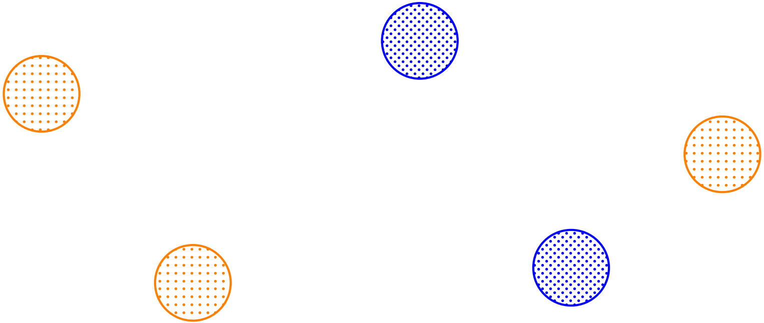

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THE NEAREST NEIGHBOR LEARNER

makes at most one mistake made per cluster

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CONVERGENCE RESULT FOR WELL-SEPARATED CLUSTERS

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CONVERGENCE RESULT FOR WELL-SEPARATED CLUSTERS

Let ν be a finite measure on $\mathcal{X}$. The nearest neighbor learner achieves vanishing error rate against any ν-dominated adversary.

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- The ν-dominated adversary selects points from $\mathcal{X}_{\text{small}}$ at rate $\mu(\mathcal{X}_{\text{small}}) < \varepsilon$.
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- Thus, the asymptotic mistake rate is upper bounded by ε almost surely.

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  - By the law of large number, at most an ε-fraction of $(x_t)_t$ comes from $\mathcal{X}_{\text{small}}$.
- Thus, the asymptotic mistake rate is upper bounded by ε almost surely.
  - Simultaneously apply upper bound for a countable collection of $\varepsilon_k \downarrow 0$.

Example: learning labels for well-separated clusters

Proof sketch.

- Split $\mathcal{X}$ into two pieces $\mathcal{X}_{\text{easy}} \cup \mathcal{X}_{\text{small}}$, where:
  - $\mathcal{X}_{\text{easy}}$ is a finite union of clusters
  - $\mathcal{X}_{\text{small}}$ contains very little mass $\nu(\mathcal{X}_{\text{small}}) < \delta$.
    - Such a decomposition exists for any $\delta > 0$ by the finiteness of ν.
- Nearest neighbor can only make finitely many mistakes on $\mathcal{X}_{\text{easy}}$.
  - These mistakes contribute nothing to the asymptotic mistake rate.
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The asymptotic mistake rate is zero. □

Generalizing the argument

The argument works even if the clusters are not well-separated.

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KEY PROPERTY USED

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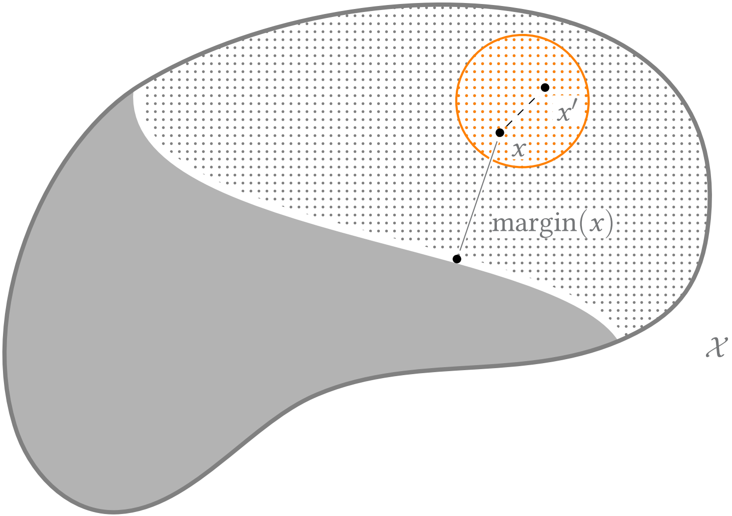
KEY PROPERTY USED

The nearest neighbor learner makes at most one mistake per mutually-labeling set.

- We introduce the device of mutually-labeling sets $U \subset \mathcal{X}$ satisfying the property:

$$\text{interpoint distances in } U < \text{distance to points with different labels.}$$

Mutually-labeling set



Generalizing argument

Definition (Mutually-labeling set)

*A subset $U \subset \mathcal{X}$ is **mutually labeling** if for all $x, x' \in U$:*

$$\underbrace{\rho(x, x')}_{\textit{interpoint distances}} < \underbrace{\mathrm{margin}(x)}_{\textit{distance to decision boundary}}$$

where $\mathrm{margin}(x)$ is the smallest distance between x and points with different labels:

$$\mathrm{margin}(x) = \inf\{\rho(x, \overline{x}) : f(x) \neq f(\overline{x})\}.$$

Convergence result

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Theorem (Convergence of nearest neighbor)

The nearest neighbor rule achieves vanishing mistake rate against a ν-dominated adversary:

$$\lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} \mathbb{1}\{\hat{y}_t \neq y_t\} = 0 \quad \text{a.s.}$$

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By prior argument, the mistake rate converges to zero almost surely. □

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QUESTIONS

- Does the ν-dominated adversary balance between generality and tractability well?
- Can the arguments still hold when there is benign label noise?
- What do meaningful rates of convergence look like?

Big picture: smoothed analysis for online learning

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OPPORTUNITY: we might not live in the worst-case adversarial setting

- Is the ν-dominated online learning setting realistic and tractable?
- If so, can we design and analyze algorithms specifically for this setting?
  - e.g. a minimax optimal algorithm might not be optimal in this setting

Thank you

ACKNOWLEDGEMENTS

Joint work with Sanjoy Dasgupta.

This work is under review, but send me an email for paper/further discussion: *agso@eng.ucsd.edu*.

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