Stochastic calculus on manifolds

Part II: Brownian motion on manifolds

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I. Characterizations of Brownian motion on $\mathbb{R}^n$

Brownian motion: as random paths

Definition (Brownian motion)

An $\mathbb{R}^n$-stochastic process $(X_t^x)_{t\geq 0}$ is ***Brownian motion*** *with $X_0 = x$ if:*

(i) *it has continuous sample paths (that is, $t \mapsto X_t$ is continuous),*

(ii) *$X_t - X_s$ for $t \geq s$ is normally distributed with mean zero and covariance $(t-s)\mathrm{I}_n$.*

In other words, we can define Brownian motion as:

- a Markov process with continuous paths and transition kernel:

$$p(t, x, y) = \left(\frac{1}{2\pi t}\right)^{n/2} \exp\left(-\frac{|x-y|^2}{2t}\right).$$

- Notation: let P^x denote the probability law over sample paths X_t^x in $C(\mathbb{R}^n)$ and let E^x be integration w.r.t. P^x.

Brownian motion: as seen by smooth functions

We can also characterize Brownian motion by the behavior of $f(X_t)$ for $f \in C^\infty(\mathbb{R}^n)$.

- Recall that this was how we motivated the **infinitesimal generator**,

$$Lf(x) := \lim_{t \downarrow 0} \frac{E^x f(X_t) - f(x)}{t} = \frac{1}{2}\Delta f(x),$$

where $\Delta = \sum \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator.

Brownian motion: as seen by smooth functions

Characterizations of Brownian motion. If for all $f \in C^\infty(\mathbb{R}^n)$, any of the following hold:

- **Dynkin's formula**

$$E^x f(X_t) = f(x) + \int_0^t Lf(X_s)\, ds.$$

- **Ito's formula**, (a 'microscopic refinement' of Dynkin's)

$$f(X_t) = f(x) + \int_0^t \nabla f(X_s)^\top dX_s + \int_0^t Lf(X_s)\, ds.$$

- Solution to the **martingale problem**, (in between Dynkin's and Ito's)

$$M_t^f := f(X_t) - f(x) - \int_0^t Lf(X_s)\, ds$$

X_t is Brownian motion if M_t^f is a martingale.

Brownian motion: as seen by smooth functions

This view of using smooth functions as *test functions* to describe Brownian motion means we can extend stochastic analysis to manifolds.

- Brownian motion X_t on a smooth manifold $\mathcal{M}$ can be described through $f(X_t)$.

II. Stochastic processes on manifolds

Semimartingales on $\mathbb{R}^\ell$

Definition

An $\mathbb{R}^\ell$-stochastic process Z_t is a ***semimartingale*** *if it has a decomposition:*

$$Z_t = M_t + A_t,$$

where M_t is a martingale and A_t is a càdlàg adapted process with bounded variation.

Recall that semimartingales are the largest class of stochastic integrators that make sense for the Ito or Stratonovich integrals:

- If Z_t is an $\mathbb{R}^\ell$-semimartingale and $V : \mathbb{R}^n \to \mathbb{R}^{n\times\ell}$ is a nice enough vector field, then:

  $$X_t = X_0 + \int_0^t V(X_s)\, dZ_s \qquad Y_t = Y_0 + \int_0^t V(Y_s) \circ dZ_s$$

  are defined, and X_t and Y_t are semimartingales.

Semimartingales on smooth manifolds

Definition

*Let $\mathcal{M}$ be a smooth manifold. A continuous, adapted $\mathcal{M}$-valued stochastic process X_t is a $\mathcal{M}$-**valued semimartingale** if $f(X_t)$ is an $\mathbb{R}$-semimartingale for all $f \in C^\infty(\mathcal{M})$.*

Ito's formula for Stratonovich SDEs

Let $V_1, \ldots, V_\ell : \mathbb{R}^n \to \mathbb{R}^n$ are smooth vector fields and Z_t an $\mathbb{R}^\ell$-semimartingale. Let X_t be the solution to the Stratonovich SDE:

$$dX_t = V_\alpha(X_s) \circ dZ_t^\alpha.$$

Then **Ito's formula** states that for $f \in C^\infty(\mathbb{R}^n)$,

$$df(X_t) = V_\alpha f(X_s) \circ dZ_t^\alpha,$$

where $V_\alpha f(x)$ is the directional derivative of f at x in the V_α direction.

- The equivalent process defined as an Ito SDE would satisfy:

$$df(X_t) = V_\alpha f(X_t) dZ_t + \frac{1}{2} \nabla_{V_\beta} V_\alpha f(X_s) d\langle Z^\alpha, Z^\beta \rangle_t.$$

SDEs on smooth manifolds

If $V_1, \ldots V_\ell \in \Gamma(T\mathcal{M})$ are smooth vector fields on $\mathcal{M}$ and Z_t an $\mathbb{R}^\ell$ semimartingale, we would like an appropriate definition for the stochastic differential equation:

$$dX_t = V_\alpha(X_t) \circ dZ_t^\alpha.$$

- We *define* an $\mathcal{M}$-semimartingale X_t to be a solution to the above SDE if it satisfies Ito's formula. That is, for all $f \in C^\infty(\mathcal{M})$,

$$df(X_t) = V_\alpha f(X_t) \circ dZ_t^\alpha.$$

Existence and uniqueness

Theorem (Hsu (2002))

There is a unique solution to $dX_t = V_\alpha(X_t) \circ dZ_t^\alpha$ *up to its explosion time.*

- Pass to Euclidean case via Whitney's embedding theorem.

Review: Riemannian manifold

Let $\mathcal{M}$ be an n-Riemannian manifold with $\langle \cdot , \cdot \rangle$ its Riemannian metric.

- The tangent space $T_p\mathcal{M}$ is the set of tangent vectors at p.
  - $T_p\mathcal{M}$ is a linear space with $\dim(T_p\mathcal{M}) = n$.
  - If $\xi = (x^1, \dots, x^n) : \mathcal{M} \to \mathbb{R}^n$ is a coordinate system at p, then $\partial_1, \dots, \partial_n$ form a basis.
- The metric describes the distance between two points p and $p + ds$.
- The Levi-Civita connection ∇ describes how to take derivatives. If $X, Y \in \Gamma(T\mathcal{M})$ are smooth vector fields, then:

$$\nabla_X Y = \text{the change of } Y \text{ along the } X \text{ direction.}$$

  - We say that a vector field Y is parallel along a curve x_t if:

$$\nabla_{\dot{x}} Y = 0.$$

- Since taking derivatives is linear, it suffices to describe ∇ using a basis. The Christoffel symbol gives ∇ in local coordinates:

$$\nabla_{\partial_i}\partial_j = \Gamma_{ij}^k \partial_k.$$

Review: orthonormal frame bundle

A frame at $p \in \mathcal{M}$ is an isomorphism $u : \mathbb{R}^n \to T_p\mathcal{M}$ is orthonormal if:

$$\left\langle ue_i, ue_j \right\rangle = \delta_{ij}.$$

- That is, u is a choice of orthonormal basis on $T_p\mathcal{M}$.
- The orthonormal frame bundle $\mathcal{O}(\mathcal{M})$ is the set of all orthonormal frames, which can be made a smooth manifold of dimension $n + n(n+1)/2$.
  - The tangent space $T_u\mathcal{O}(\mathcal{M})$ can be decomposed:

    $$T_u\mathcal{O}(\mathcal{M}) = \mathcal{H}_u\mathcal{O}(\mathcal{M}) \oplus \mathcal{V}_u\mathcal{O}(\mathcal{M}),$$

    where $\mathcal{H}_u\mathcal{O}(\mathcal{M})$ are the horizontal vectors corresponding to moving tangent along $\mathcal{M}$ and $\mathcal{V}_u\mathcal{O}(\mathcal{M})$ are the vertical vectors corresponding to rotating the frame.
  - Given a vector field X on $\mathcal{M}$ there is a unique horizontal vector fields X^*, called its horizontal lift, such that $\pi_* X^* = X$ where $\pi : \mathcal{O}(\mathcal{M}) \to \mathcal{M}$ is the projection map.
- Given a smooth curve x_t and initial frame u_0 at x_0, the horizontal lift of x_t is the unique curve u_t in $\mathcal{O}(\mathcal{M})$ such that for any $v \in \mathbb{R}^n$, $u_t v$ is parallel along x_t.

Example: one-dimensional case

Let $\mathcal{M} = \mathbb{R}$ with the usual (global) coordinate system.

- Let ∇ be a connection on $\mathcal{M}$ with:

$$\nabla_{\partial}\partial(x) = \Gamma(x)\partial.$$

- Consider the curve $x_t = t$. Let $u_0 s = s\partial$ be the initial frame. What is the horizontal lift of x_t with initial frame u_0?

Example: one-dimensional case

To describe a frame u, we just need to specify (x, u) the point $x \in \mathcal{M}$ and the basis vector $u\partial$ (we've overloaded the notation u to denote the frame and the coordinate).

- Under this choice of coordinates, $u_0 \mapsto (0, 1)$.
- The condition that u_t is parallel to itself is to say:

$$0 = \nabla_\partial (u_t \partial) = u_t \Gamma(t) + \partial(u_t),$$

where $\partial(u_t) = \dot{u}_t$. This implies u_t is described by the coordinates (t, u_t), where:

$$u_t = \exp\left(- \int_0^t \Gamma(s)\, ds \right).$$

Review: anti-development

Given a smooth curve x_t on $\mathcal{M}$, let u_t be the horizontal lift with initial frame u_0.

- Notice that $u_t : \mathbb{R}^n \to T_{x_t}\mathcal{M}$ and $\dot{x}_t \in T_{x_t}\mathcal{M}$.
- The **anti-development** of the curve x_t is the curve w_t in $\mathbb{R}^n$,

$$w_t = \int_0^t u_s^{-1} \dot{x}_s \, ds.$$

- Recall the spaceship+joystick analogy. Then w_t is the curve that the joystick traces out to move the spaceship on the curve x_t.
  - The **development** of a curve w_t in $\mathbb{R}^n$ is the reverse process to construct x_t (the spaceship develops the input to its joystick into a path in space).

Example: one-dimensional case

Returning to our earlier example, what is the development of the curve $x_t = t$?

- Since $\dot{x}_t = \partial$, we have:

$$u_t^{-1}\dot{x}_t = \exp\big(G(t)\big),$$

  where $G(t) = \int_0^t \Gamma(s)\, ds$.

- It follows that the anti-development is:

$$w_t = \int_0^t e^{G(s)}\, ds.$$

- Checking with intuition: suppose $\Gamma \equiv 1$, so that:

$$\nabla_\partial \partial = 1.$$

  - Suppose you're travelling along the curve x_t and to go unit speed, you need to push the joystick forward some amount. To maintain that speed, you need to push the joystick more proportional to the amount it is already pushed. Indeed, $w_t = e^t$.

Semimartingales on the frame bundle

Let $u \in \mathcal{O}(\mathcal{M})$ be a frame, so that $u : \mathbb{R}^n \to T_p\mathcal{M}$.

- Let H_i be the horizontal vector field where $H_i(u)$ is the lift of $ue_i \in T_p\mathcal{M}$ at u.
- In the following, we consider the SDE on the frame bundle:

$$dU_t = \sum_{i=1}^{n} H_i(U_t) \circ dW_t^i = H(U_t) \circ dW_t.$$

Extending horizontal lifts and developments to semimartingales

> “As expected, for semimartingales stochastic development and horizontal lift are obtained by solving stochastic differential equations driven by either $\mathbb{R}$-valued or $\mathcal{M}$-valued semimartingales. **But unlike the case of smooth curves, these equations are not local at a fixed time.**

Hsu (2002), p. 44

Stochastic anti-developments and horizontal lifts

Definition (Anti-development)

An $\mathcal{O}(\mathcal{M})$-semimartingale U_t is ***horizontal*** *if there exists semimartingale W_t such that:*

$$dU_t = H(U_t) \circ dW_t.$$

The unique W_t is called the ***anti-development*** *of U_t.*

Definition (Development)

Given W_t, the solution U_t to the above SDE is called the ***development*** *of W_t.*

Definition (Horizontal lift)

Let X_t be a $\mathcal{M}$-semimartingale. An $\mathcal{O}(\mathcal{M})$-semimartingale U_t is a ***horizontal lift*** *of X_t if its projection is X_t,*

$$\pi U_t = X_t.$$

The correspondences

$$W \in \mathbb{R}^n \underset{\text{anti-development}}{\overset{\text{development}}{\rightleftharpoons}} U \in \mathcal{O}(\mathcal{M}) \underset{\text{horizontal lift}}{\overset{\text{projection}}{\rightleftharpoons}} X \in \mathcal{M}$$

- *development* from existence + uniqueness theorem for SDEs on manifolds
- *projection* from $\pi : \mathcal{O}(\mathcal{M}) \to \mathcal{M}$

For the remaining correspondences, assume that $\mathcal{M}$ is a closed submanifold of $\mathbb{R}^N$ (note that Nash's embedding theorem states that it is always possible to isometrically embed a Riemannian manifold into Euclidean space).

- We end up with an anti-development W in $\mathbb{R}^N$ for some N possibly larger than n.
- A construction with local charts possible, but technically unwieldy (Hsu, 2008).

Stochastic anti-development and horizontal lift

Theorem (Theorem 2.3.4, Hsu (2002))

*A horizontal semimartingale U_t on the frame bundle $\mathcal{O}(\mathcal{M})$ has a **unique anti-development** W_t.*

Theorem (Theorem 2.3.5, Hsu (2002))

*Suppose that X_t is a semimartingale on $\mathcal{M}$ and $U_0 \in \mathcal{O}(\mathcal{M})$ with $\pi U_0 = X_0$. Then there is a **unique horizontal lift** U_t of X_t starting from U_0.*

Example: one-dimensional case

- The horizontal lift of X_t is given by:

$$U_t = \left(X_t, e^{-G(X_t)}\right),$$

where $G(x) = \int_0^x \Gamma(s)\, ds$.

- The anti-development of U_t is:

$$W_t = \int_0^t e^{G(X_s)}\, ds.$$

The correspondence in equations

Proposition (Proposition 2.3.8, Hsu (2002))

Let X_t be a $\mathcal{M}$-semimartingale with initial condition X_0 satisfying:

$$dX_t = V_\alpha(X_t) \circ dZ_t^\alpha.$$

Let V_α^* be the horizontal lift of V_α to $\mathcal{O}(\mathcal{M})$. Then:

(i) *the horizontal lift U_t of X_t with initial frame U_0 is a solution to:*

$$dU_t = V_\alpha^*(U_t) \circ dZ_t^\alpha,$$

(ii) *the anti-development of X_t is given by:*

$$W_t = \int_0^t U_s^{-1} V^\alpha(X_s) \circ dZ_\alpha^t.$$

Eells-Elworhthy-Malliavin construction of Brownian motion

If W_t is Brownian motion on $\mathbb{R}^n$, then its development U_t which is the solution to:

$$dU_t = H_\alpha(U_t) \circ dW_t^\alpha$$

is *horizontal Brownian motion* on $\mathcal{O}(\mathcal{M})$.

- The projection $X_t = \pi U_t$ is **Brownian motion** on $\mathcal{M}$.

Martingales on manifolds

Let $\mathcal{M}$ be a Riemannian manifold equipped with the Levi-Civita connection ∇.

Definition

*An $\mathcal{M}$-valued semimartingale X_t is a ∇-**martingale** if its anti-development W_t with respect to ∇ is a (local) martingale.*

Characterization of martingales

Proposition (Proposition 2.5.2, Hsu (2002))

An $\mathcal{M}$-semimartingale X_t is a ∇-martingale if and only if for $f \in C^\infty(\mathcal{M})$:

$$M_t^f := f(X_t) - f(X_0) - \int_0^t \frac{1}{2}\nabla^2 f(X_s)(dX_s, dX_s)$$

is a local martingale (in the usual sense).

Proof of proposition

- $(\Rightarrow)$. Show that if X_t is a ∇-martingale, then M_t^f is a martingale .
  - Lift X_t to U_t and lift $f : \mathcal{M} \to \mathbb{R}$ to $\tilde{f} : \mathcal{O}(\mathcal{M}) \to \mathbb{R}$.
  - Apply Ito's formula to $\tilde{f}(U_t)$.
- $(\Leftarrow)$. Show that if M_t^f is a martingale for all smooth f, then X_t is a ∇-martingale.
  - Assume $\mathcal{M}$ is embedded into Euclidean space.
  - Consider f to be the coordinate function $f(x) = x$.
  - Claim: the anti-development W_t can be computed as:

$$W_t = \int_0^t V_s dM_t^f ,$$

    for some process V_s. If M_t^f is a local martingale, then so is W_t.

$\square$

Diffusion processes

Let L be a smooth second-order elliptic (not necessarily non-degenerate) operator on $\mathcal{M}$.

Definition

*An adapted stochastic process X_t is called a L-**diffusion process** if X_t is an $\mathcal{M}$-valued semimartingale and:*

$$M_t^f = f(X_t) - f(X_0) - \int_0^t Lf(X_s)\, ds$$

is a local martingale for all $f \in C^\infty(\mathcal{M})$.

Gradients, divergences, and the Laplace-Beltrami operator

Given a Riemannian manifold, define the following:

- the **gradient** $\operatorname{grad} f$ of a function is the dual of the differential df
- the **divergence** $\operatorname{div} X$ of a vector field is the contraction of the $(1,1)$-tensor ∇X.
- the **Laplace-Beltrami operator** $\Delta_{\mathcal{M}}$ is defined:

$$\Delta_{\mathcal{M}} = \operatorname{div} \operatorname{grad} f.$$

Proposition

For any orthonormal basis $\{b_i\}$ of $T_p\mathcal{M}$,

$$\Delta_{\mathcal{M}} f = \operatorname{trace} \nabla^2 f = \sum_{i=1}^{n} \nabla^2 f(b_i, b_i).$$

Bochner's horizontal Laplacian

Recall that H_i are the horizontal vector fields where $H_i(u)$ is the lift of $ue_i \in T_p\mathcal{M}$ at u.

- The **Bochner's horizontal Laplacian** is defined:

$$\Delta_{\mathcal{O}(\mathcal{M})} = \sum_{i=1}^{n} H_i^2.$$

Proposition

The Bochner's horizontal Laplacian $\Delta_{\mathcal{O}(\mathcal{M})}$ is the lift of the Laplace-Beltrami operator $\Delta_{\mathcal{M}}$ to $\mathcal{O}(\mathcal{M})$. That is, if $f \in C^\infty(\mathcal{M})$, let $\tilde{f}$ be the lift to $\mathcal{O}(\mathcal{M})$. Then:

$$\Delta_{\mathcal{M}} f(x) = \Delta_{\mathcal{O}(\mathcal{M})} \tilde{f}(u),$$

where $\pi u = x$.

Characterization of Brownian motion

Proposition (Proposition 3.2.1, Hsu (2002))

Let X_t be an $\mathcal{M}$-semimartingale. The following are equivalent:

(i) *X_t is a (Riemannian) Brownian motion.*

(ii) *X_t is an $\frac{1}{2}\Delta_{\mathcal{M}}$-diffusion process. That is, for all $f \in C^\infty(\mathcal{M})$,*

$$M_t^f = f(X_t) - f(X_0) - \int_0^t \frac{1}{2}\Delta_{\mathcal{M}} f(X_s)\, ds$$

is a local martingale.

(iii) *The anti-development W_t of X_t is standard Euclidean Brownian motion.*

Characterization of horizontal Brownian motion

Proposition (Proposition 3.2.2, Hsu (2002))

Let U_t be an $\mathcal{O}(\mathcal{M})$-semimartingale. The following are equivalent:

(i) *U_t is a horizontal Brownian motion on $\mathcal{O}(\mathcal{M})$.*

(ii) *U_t is an $\frac{1}{2}\Delta_{\mathcal{O}(\mathcal{M})}$-diffusion process.*

(iii) *U_t is horizontal whose projection $X_t = \pi U_t$ is Brownian motion on $\mathcal{M}$.*

(iv) *U_t is horizontal whose anti-development is standard Euclidean Brownian motion.*

References

Elton P Hsu. *Stochastic analysis on manifolds.* Number 38. American Mathematical Soc., 2002.

Elton P Hsu. A brief introduction to Brownian motion on a Riemannian manifold. *Lecture Notes*, 2008.