Stochastic differential equations

Basic results

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Ito stochastic differential equation

Definition

Let $(B_t)_{t\geq 0}$ be Brownian motion. A stochastic process $(X_t)_{t\geq 0}$ is a solution to the ***Ito stochastic differential equation***

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t$$

if for all $t \geq 0$ and X_0,

$$X_t = X_0 + \int_0^t b(s, X_s)\, ds + \int_0^t \sigma(s, X_s)\, dB_s.$$

We call any such solution an ***Ito process****.*

Existence and uniqueness

Conditions from ODEs

Question: what conditions can we impose on b and σ to ensure that there exists a unique solution to the following SDE:

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t$$

Counterexample: existence

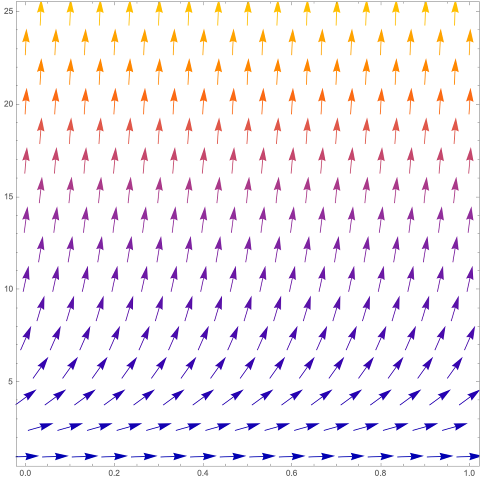

Figure 1: The solution for the ODE $dX_t = X_t^2 dt$ and $X_0 = 1$ explodes.

Counterexample: existence

The following ODE does not have a global solution:

$$\frac{dX_t}{dt} = X_t^2 \quad \text{and} \quad X_0 = 1.$$

It has the unique solution for $t \in [0, 1)$.

$$X_t = \frac{1}{1-t}.$$

- Introducing a **growth condition** $|b(t,x)| \le K(1+|x|)$ ensures that the solution to the ODE does not *explode*.[1]

[1] We denote the norm of $x \in \mathbb{R}^n$ by $|x|$.

Counterexample: uniqueness

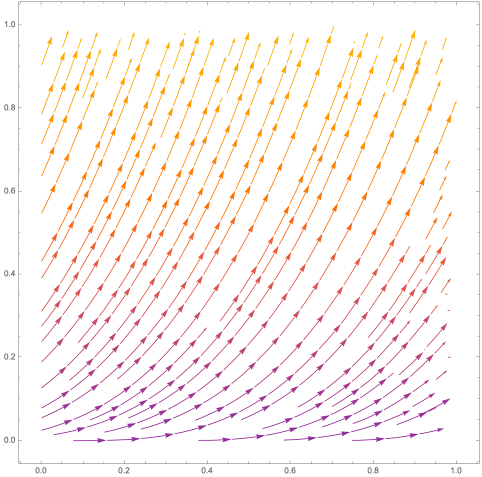

Figure 2: Streamlines of the ODE $dX_t = 3X_t^{2/3}\,dt$.

Counterexample: uniqueness

The following ODE has more than one solution:

$$\frac{dX_t}{dt} = 3X_t^{2/3} \quad \text{and} \quad X_0 = 0.$$

For any $a > 0$, the following function is a solution:

$$X_t = \begin{cases} 0 & t \leq a \\ (t-a)^3 & t > a. \end{cases}$$

- Introducing a **Lipschitz condition** $|b(t,x) - b(t,y)| \leq L|x-y|$ ensures that any solution to the ODE is unique.

An existence and uniqueness result for SDEs

Theorem (Theorem 5.2.1, Øksendal (2003))

Let $T > 0$. *Let* $b : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ *and* $\sigma : [0, T] \times \mathbb{R}^n \to \mathbb{R}^{n \times m}$ *be measurable.*
Suppose that b *and* σ *satisfy:*

- *growth condition:* $|b(t, x)| + |\sigma(t, x)| \leq K(1 + |x|)$
- *Lipschitz condition:* $|b(t, x) - b(t, y)| + |\sigma(t, x) - \sigma(t, y)| \leq L|x - y|$.

Further, suppose Z *and* $(B_t)_{t \geq 0}$ *are independent with* $E[|Z|^2] < \infty$. *Then, the SDE*

$$dX_t = b(t, X_t)\,dt + \sigma(t, X_t)\,dB_t$$

with $X_0 = Z$ *on the time interval* $t \in [0, T]$ *has a unique* t*-continuous solution* $X_t(\omega)$ *such that* $X_t(\omega)$ *is adapted to the filtration* $\mathcal{F}_t^Z$ *generated by* Z *and* $(B_s)_{0 \leq s \leq t}$, *and such that:*

$$E\left[\int_0^T |X_t|^2 \, dt\right] < \infty.$$

Uniqueness of solutions

Givens: $dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t$ with:

- Lipschitz condition: $|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \leq L|x-y|$.

Proof idea: Let X_t and $\widehat{X}_t$ be solutions.

- Apply *Ito isometry* and *Lipschitz condition* to show that:

$$E\left[|X_t - \widehat{X}_t|^2\right] = 0.$$

- This implies that for any $t \geq 0$,

$$P\left[|X_t - \widehat{X}_t| = 0\right] = 1.$$

- This simultaneously holds for countable number of $t \in \mathbb{Q} \cap [0, T]$.
- Uniqueness follows from continuity of $t \mapsto |X_t - \widehat{X}_t|$.

Existence of solutions

Givens: $dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t$ with:

- growth condition: $|b(t,x)| + |\sigma(t,x)| \leq K(1+|x|)$
- Lipschitz condition: $|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \leq L|x-y|$.

Proof idea:

- Construct sequence $\left(Y_t^{(k)}\right)_{k\in\mathbb{N}}$ inductively by:

$$Y_t^{(0)} = X_0 \quad \text{and} \quad Y_t^{(k+1)} = X_0 + \int_0^t b(s, Y_s^{(k)})dt + \int_0^t \sigma(s, Y_s^{(k)}\, dB_s.$$

- Show that $\left(Y_t^{(k)}\right)_{k\in\mathbb{N}}$ is a Cauchy sequence in $L^2([0,T] \times \Omega; \lambda \times P)$.
- Set $X_t = \lim\limits_{k\to\infty} Y_t^{(k)}$ and show that X_t satisfies SDE and can be continuous.

Solutions of SDEs are Markov process

Markov processes

Definition

A stochastic process $(X_t)_{t\geq 0}$ is ***Markov*** *with respect to a filtration $(\mathcal{F}_t)_{t\geq 0}$ if*

- *X_t is adapted to the filtration, and*
- *for any $s > t$, the conditional independence holds, $X_s \perp\!\!\!\perp \mathcal{F}_t | X_t$.*

Solutions are Markov

Proposition

Let $(B_t)_{t\geq 0}$ be Brownian motion and let $(\mathcal{F}_t)_{t\geq 0}$ be its natural filtration. Let $(X_t)_{t\geq 0}$ be a solution to an SDE satisfying the conditions of the existence/uniqueness theorem,

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t.$$

Then, $(X_t)_{t\geq 0}$ is Markov with respect to $(\mathcal{F}_t)_{t\geq 0}$.

Change of variables: Ito's formula

The infinitesimal: $dB_t^2 = dt$ and $dB_t^{2+N} = 0$

Recall that we can think of dB_t as an *infinitesimal of order* $\frac{1}{2}$, in the sense:

$$\int_0^T f(t,\omega)\, dB_t(\omega)^2 = \int_0^T f(t,\omega)\, dt \quad \text{and} \quad \int_0^T f(t,\omega)\, dB_t(\omega)^{2+N} = 0 \quad (N > 0),$$

for arbitrary adapted function f. Here, the integral is defined:

$$\int_0^T f(t,\omega)\, dB_t(\omega)^{2+N} = \lim_{n\to\infty} \sum_{\Pi_n} f(t_{i-1},\omega)\, [B_{t_i}(\omega) - B_{t_{i-1}}(\omega)]^{2+N},$$

where Π_n is a partition $0 = t_0 < t_1 < \cdots < t_n = T$ such that $\lim_{n\to\infty} \|\Pi_n\| = 0$.
Recall that $\|\Pi_n\| = \max_k\, t_k - t_{k-1}$.

Ito formula intuition

Let $dX_t = u(t, X_t)\, dt + v(t, X_t)\, dB_t$. Consider the Taylor expansion:

$$\begin{aligned} df(X_t) &= f(X_t + dX_t) - f(X_t) \\ &= f(X_t) dX_t + \frac{1}{2} f''(X_t) dX_t^2 + \cdots \\ &= f'(X_t) \left\{ u\, dt + v\, dB_t \right\} + \frac{1}{2} f''(X_t)\, v^2\, dB_t^2, \end{aligned}$$

where all higher-order terms have been discarded. Apply $dB_t^2 = dt$ to obtain:

$$df(X_t) = \left\{ uf' + \frac{1}{2} v^2 f'' \right\} dt + vf'\, dB_t.$$

Ito formula

Theorem (1-dimensional Ito formula, Øksendal (2003))

Let X_t be an Ito process given by $dX_t = u(t, X_t)\, dt + v(t, X_t)\, dB_t$. Let $Y_t = f(t, X_t)$ where f is twice continuously differentiable. Then Y_t is an Ito process given by:

$$dY_t = \frac{\partial f}{\partial t}(t, X_t)dt + \frac{\partial f}{\partial x}(t, X_t)dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, X_t)dX_t^2$$

Ito formula

Theorem (The general Ito formula, Øksendal (2003))

Let X_t be an Ito process given by $dX_t = u(t, X_t)\, dt + v(t, X_t)\, dB_t$. Let $Y_t = f(t, X_t)$ where $f : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^p$ is C^2. Then Y_t is an Ito process whose kth component $Y_t^{(k)}$ is given by:

$$dY_t^{(k)} = \frac{\partial f_k}{\partial t}(t, X_t)dt + \sum_{i=1}^{n} \frac{\partial f_k}{\partial x_i}(t, X_t)dX_t^{(i)} + \frac{1}{2}\sum_{i,j} \frac{\partial^2 f_k}{\partial x_i \partial x_j}(t, X_t)dX_i dX_j,$$

where $dB_i\, dB_j = \delta_{ij} dt$ and $dB_i\, dt = dt\, dB_i = 0$.

Application: Ornstein-Uhlenbeck process

Ornstein-Uhlenbeck process

Definition

*The **Ornstein-Uhlenbeck process** is a stochastic process $(X_t)_{t\geq 0}$ satisfying the Ito SDE:*

$$dX_t = -kX_t dt + \sqrt{D} dB_t.$$

- We can think of it as a continuous random walk that tends to revert back to zero.
- A physical example is a spring with thermal fluctuations.

Change of variables

Let $(X_t)_{t\geq 0}$ be the Ornstein-Uhlenbeck process, so that $dX_t = -kX_t dt + \sqrt{D} dB_t$.
Set $Y_t = X_t e^{kt}$. Apply the Ito formula:

$$\begin{aligned} dY_t &= \frac{\partial f}{\partial t}(t, X_t) dt + \frac{\partial f}{\partial x}(t, X_t) dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, X_t) dX_t^2 \\ &= kX_t e^{kt} dt + e^{kt} dX_t \\ &= kX_t e^{kt} dt - kX_t e^{kt} dt + \sqrt{D} e^{kt} dB_t. \end{aligned}$$

Therefore, $d(X_t e^{kt}) = \sqrt{D} e^{kt} dB_t$.

Integrate

Integrating the SDE $d(X_t e^{kt}) = \sqrt{D} e^{kt} dB_t$, we obtain:

$$X_t = X_0 e^{-kt} + \sqrt{D} \int_0^t e^{-k(t-s)} dB_s.$$

- If X_0 is Gaussian or deterministic, then X_t is Gaussian,

$$\mathbb{E}[X_t] = \mathbb{E}[X_0] e^{-kt} \quad \text{and} \quad \mathrm{Var}[X_t] = \mathrm{Var}[X_0] e^{-2kt} + \frac{D}{2k},$$

  by independence of the two terms of X_t and Ito isometry.[2]

[2] See also (Gardiner, 1985).

References

Crispin W Gardiner. *Handbook of stochastic methods.* Springer Berlin, 1985.

Bernt Øksendal. *Stochastic differential equations: an introduction with applications.* Springer, 2003.